Problema săptămânii 402

Fie a, b, c numere reale pozitive astfel încât $a^2 + b^2 + c^2 + ab + ac + bc = 6$. Arătaţi că

$$\sqrt{a^2 + 3b^2} + \sqrt{b^2 + 3c^2} + \sqrt{c^2 + 3a^2} \geq 6.$$

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Soluţie: Prin ridicare la pătrat, inegalitatea de demonstrat se scrie echivalent
$4(a^2 + b^2 + c^2) + 2\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + 2\sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} +$
$2\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)}) \geq 36$, adică, folosind condiţia din enunţ,
$4(a^2 + b^2 + c^2) + 2\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + 2\sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} +$
$2\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)} \geq 6(a^2 + b^2 + c^2 + ab + bc + ca)$.
Această ultimă inegalitate revine la $\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + \sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} +$
$\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)} \geq a^2 + b^2 + c^2 + 3ab + 3bc + 3ca$.
Ea rezultă din inegalitatea Cauchy-Buniakowski-Schwarz.
Avem $(a^2+3b^2)(b^2+3c^2) = (a^2+2b^2+b^2)(c^2+2c^2+b^2) \geq (ac+2bc+b^2)^2$ şi analoagele, deci $\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + \sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} + \sqrt{(c^2 + 3a^2)(a^2 + 3b^2)} \geq$
$(ac + 2bc + b^2) + (ba + 2ca + c^2) + (cb + 2ab + a^2) = a^2 + b^2 + c^2 + 3(ab + bc + ca)$.
Egalitatea are loc dacă şi numai dacă $a = b = c = 1$.

Am primit soluţii de la: *Titu Zvonaru* şi .

Problem of the week no. 402

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + ab + ac + bc = 6$. Prove that

$$\sqrt{a^2 + 3b^2} + \sqrt{b^2 + 3c^2} + \sqrt{c^2 + 3a^2} \geq 6.$$

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Solution: By squaring, our inequality can be written equivalently
$4(a^2 + b^2 + c^2) + 2\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + 2\sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} +$
$2\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)}) \geq 36$, i.e., using the given condition,
$4(a^2 + b^2 + c^2) + 2\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + 2\sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} +$
$2\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)} \geq 6(a^2 + b^2 + c^2 + ab + bc + ca)$.
The last inequality comes down to $\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)} + \sqrt{(b^2 + 3c^2)(c^2 + 3a^2)} +$
$\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)} \geq a^2 + b^2 + c^2 + 3ab + 3bc + 3ca$.
This follows from the Cauchy-Buniakowski-Schwarz inequality:
we have $(a^2+3b^2)(b^2+3c^2) = (a^2+2b^2+b^2)(c^2+2c^2+b^2) \geq (ac+2bc+b^2)^2$ and its analogues, hence $\sqrt{(a^2 + 3b^2)(b^2 + 3c^2)}+\sqrt{(b^2 + 3c^2)(c^2 + 3a^2)}+\sqrt{(c^2 + 3a^2)(a^2 + 3b^2)} \geq$
$(ac + 2bc + b^2) + (ba + 2ca + c^2) + (cb + 2ab + a^2) = a^2 + b^2 + c^2 + 3(ab + bc + ca)$.
Equality holds if and only if $a = b = c = 1$.