Problema săptămânii 394

Fie a, b, c numere reale pozitive astfel încât $a + b + c = 3$. Arătaţi că

$$\frac{a^2}{b^2+1} + \frac{b^2}{c^2+1} + \frac{c^2}{a^2+1} \geq \frac{3}{2}.$$

Soluţie: Conform inegalităţii CBS (forma Titu Andreescu), avem

$$\frac{a^2}{b^2+1} + \frac{b^2}{c^2+1} + \frac{c^2}{a^2+1} = \frac{a^4}{a^2b^2+a^2} + \frac{b^4}{b^2c^2+b^2} + \frac{c^4}{c^2a^2+c^2} \geq$$

$$\geq \frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2}.$$

Aşadar, este suficient să demonstrăm că $\dfrac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2} \geq \dfrac{3}{2}$, $\qquad (*)$
adică, folosind $a + b + c = 3$,

$$2(a^2+b^2+c^2)^2 \geq 3(a^2b^2+b^2c^2+c^2a^2) + 3(a^2+b^2+c^2) \cdot \left(\frac{a+b+c}{3}\right)^2.$$

După calcule, ultima inegalitate revine la

$$5(a^4+b^4+c^4)+a^2b^2+b^2c^2+c^2a^2 \geq 2(a^3b+ab^3+b^3c+bc^3+c^3a+ca^3)+2(a^2bc+ab^2c+abc^2).$$

Această inegalitate se poate demonstra folosind inegalitatea lui Muirhead:
avem $2 \cdot [4,0,0] \geq 2 \cdot [3,1,0]$, $\frac{1}{2} \cdot [4,0,0] \geq \frac{1}{2} \cdot [2,1,1]$ şi $\frac{1}{2} \cdot [2,2,0] \geq \frac{1}{2} \cdot [2,1,1]$ care, adunate, dau inegalitatea din enunţ.
Alternativ, avem $a^4 + b^4 \geq a^3b + ab^3 \Leftrightarrow (a^3 - b^3)(a - b) \geq 0$ care este adevărată fiindcă cele două paranteze au acelaşi semn. Adunând cu analoagele şi înmulţind cu 2, avem $4(a^4 + b^4 + c^4) \geq 2(a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3)$. De asemenea, $a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2 \geq 2a^2b^2+2b^2c^2+2c^2a^2 \geq 2a^2bc+2ab^2c+2abc^2$, unde s-a folosit inegalitatea $x^2 + y^2 + z^2 \geq xy + yz + zx$ de două ori, mai întâi pentru $x = a^2$, $y = b^2$, $z = c^2$, apoi pentru $x = ab$, $y = bc$, $z = ca$.
Egalitatea are loc dacă şi numai dacă $a = b = c = 1$.

Finalizare mai simplă: Inegalitatea $(*)$ se obţine adunând inegalităţile

$$(a^2+b^2+c^2)^2 \geq 3(a^2b^2+b^2c^2+c^2a^2) \text{ şi } (a^2+b^2+c^2)^2 \geq 3(a^2+b^2+c^2)$$

care sunt imediate.

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Problem of the week no. 394

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2}{b^2+1}+\frac{b^2}{c^2+1}+\frac{c^2}{a^2+1} \geq \frac{3}{2}.$$

Solution: By Titu's Lemma, we have

$$\frac{a^2}{b^2+1}+\frac{b^2}{c^2+1}+\frac{c^2}{a^2+1}=\frac{a^4}{a^2b^2+a^2}+\frac{b^4}{b^2c^2+b^2}+\frac{c^4}{c^2a^2+c^2} \geq$$

$$\geq \frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2}.$$

Thus, it is sufficient to prove that $\dfrac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2} \geq \dfrac{3}{2}$, i.e., using the condition $a+b+c=3$, to prove that

$$2(a^2+b^2+c^2)^2 \geq 3(a^2b^2+b^2c^2+c^2a^2)+3(a^2+b^2+c^2)\cdot\left(\frac{a+b+c}{3}\right)^2.$$

After computations, this inequality comes down to

$$5(a^4+b^4+c^4)+a^2b^2+b^2c^2+c^2a^2 \geq 2(a^3b+ab^3+b^3c+bc^3+c^3a+ca^3)+2(a^2bc+ab^2c+abc^2).$$

We can prove this by making use of Muirhead's inequality:
we have $2\cdot[4,0,0] \geq 2\cdot[3,1,0]$, $\frac{1}{2}\cdot[4,0,0] \geq \frac{1}{2}\cdot[2,1,1]$ and $\frac{1}{2}\cdot[2,2,0] \geq \frac{1}{2}\cdot[2,1,1]$; summing them gives the desired inequality.
Alternatively, we have $a^4+b^4 \geq a^3b+ab^3 \Leftrightarrow (a^3-b^3)(a-b) \geq 0$ which is true because the two parantheses have the same sign. Summing these inequalities and multiplying by 2, we get $4(a^4+b^4+c^4) \geq 2(a^3b+ab^3+b^3c+bc^3+c^3a+ca^3)$.
Also, $a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2 \geq 2a^2b^2+2b^2c^2+2c^2a^2 \geq 2a^2bc+2ab^2c+2abc^2$, where we have used the inequality $x^2+y^2+z^2 \geq xy+yz+zx$ twice, first for $x=a^2$, $y=b^2$, $z=c^2$, then for $x=ab$, $y=bc$, $z=ca$.
The equality holds if and only if $a=b=c=1$.