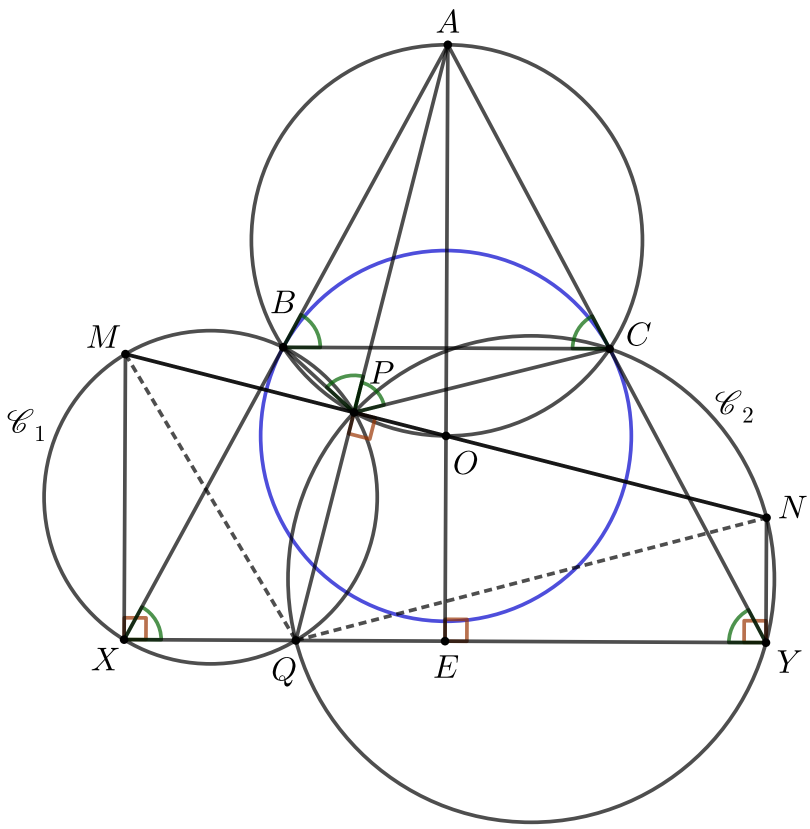
Problema săptămânii 381

Un cerc de centru O este tangent laturilor unghiului $\sphericalangle BAC$ în punctele B şi C, iar Q este un punct în interiorul unghiului $\sphericalangle BAC$. Dacă $P \in (AQ)$ este astfel încât $AQ \perp OP$, iar dreapta OP intersectează a doua oară cercurile circumscrise triunghiurilor BPQ şi CPQ în punctele M, respectiv N, demonstraţi că $OM = ON$.

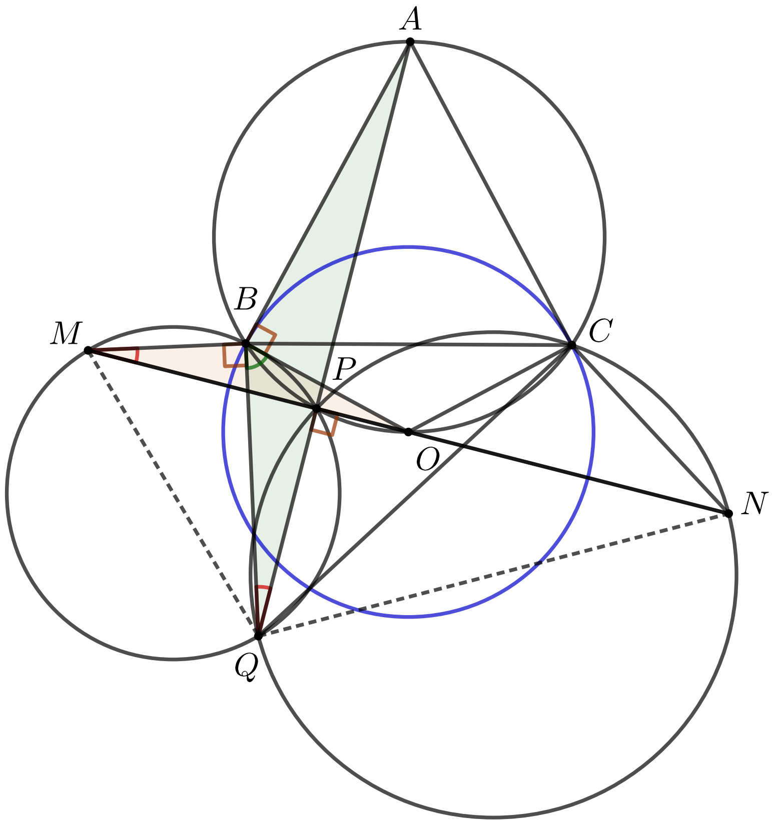
Olimpiadă Rusia, 2008

Soluţia 1: (*Cristian Muth*)
Fie X al doilea punct de intersecţie al cercului $\mathscr{C}_1$ circumscris lui BPQ cu dreapta AB, iar Y al doilea punct de intersecţie al cercului $\mathscr{C}_2$ circumscris triunghiului CPQ cu AC. Cum $MN \perp PQ$, punctele M şi N sunt diametral opuse lui Q în cercurile $\mathscr{C}_1$ şi $\mathscr{C}_2$, prin urmare $MX \perp XQ$ şi $NY \perp YQ$.
Cum A se află pe axa radicală a cercurilor $\mathscr{C}_1$ şi $\mathscr{C}_2$, avem $AB \cdot AX = AP \cdot AQ = AC \cdot AY$. Cum $AB = AC$, rezultă $AX = AY$, deci $XY \parallel BC$. Dar punctele B, P, C se află pe cercul de diametru $[AO]$, deci $ABPC$ este inscriptibil. Cum şi patrulaterele $BXQP$ şi $CYQP$ sunt inscriptibile, avem $\sphericalangle XQP = \sphericalangle ABP = 180^\circ - \sphericalangle ACP = \sphericalangle YQP$, deci punctele X, Q, Y sunt coliniare.

Dacă E este mijlocul lui $[XY]$, atunci AE este mediatoarea lui $[XY]$, deci şi bisectoarea $\sphericalangle BAC$. Astfel, $O \in AE$. Dar atunci OE este paralela dusă prin mijlocul laturii $[XY]$ la bazele trapezului dreptunghic $MXYN$, deci este linia mijlocie a trapezului, astfel că $OM = ON$.



Soluţia 2: (user *77ant* pe AoPS)
Patrulaterul $MBPQ$ fiind inscriptibil, avem $\sphericalangle AQB = \sphericalangle PQB = \sphericalangle PMB = \sphericalangle OMB$ şi $\sphericalangle MBO = \sphericalangle MPQ = 90^\circ = \sphericalangle ABO$, deci $\sphericalangle MBO = 90^\circ + \sphericalangle OBQ = \sphericalangle ABQ$. Deducem că triunghiurile ABQ şi OBM sunt asemenea. Analog se arată şi că $\Delta ACQ \sim \Delta NCO$, astfel că $\dfrac{OM}{OB} = \dfrac{AQ}{AB} = \dfrac{AQ}{AC} = \dfrac{ON}{OC}$. Cum $OB = OC$, obţinem $OM = ON$.



Remarcă: Afirmaţia din problemă rămâne adevărată şi dacă $Q \in (AP)$.

Am primit soluţii de la: *Ioana Vlădoiu, Cristian Muth, Maria Ciucu, Eric-Dimitrie Cismaru* şi *Maria Gorgan*.

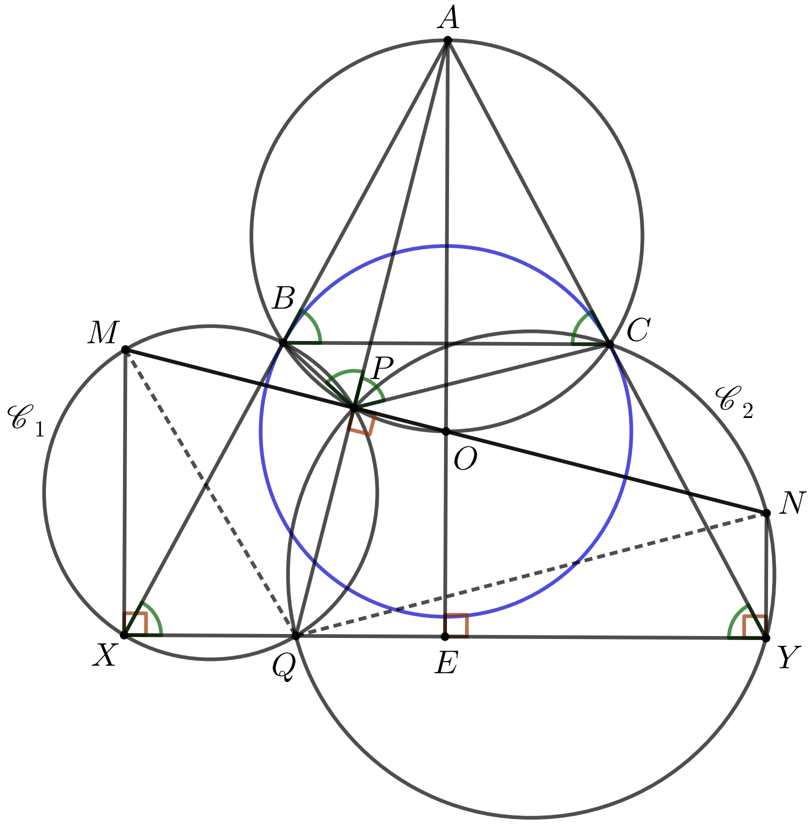
Problem of the week no. 381

A circle with center O is tangent to the sides of angle BAC at B and C. Point Q is taken inside the angle BAC. Assume that point P on the line segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles of triangles BPQ and CPQ again at points M and N, respectively. Prove that $OM = ON$.

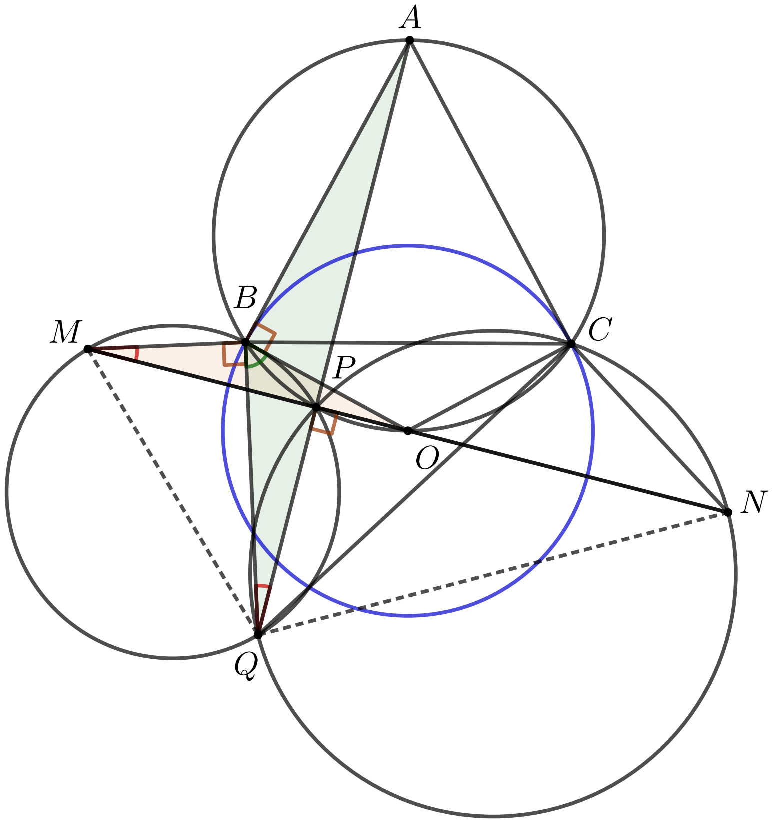
All Russian MO, 2008

Solution 1: (*Cristian Muth*)
Let X be te second intersection point of the circumcircle, $\mathscr{C}_1$, of BPQ with the line AB, and let Y be the second intersection point of te circumcircle $\mathscr{C}_2$ of CPQ with line AC. As $MN \perp PQ$, points M and N are the antipodes of Q in $\mathscr{C}_1$ and $\mathscr{C}_2$, therefore $MX \perp XQ$ and $NY \perp YQ$.
As A is on the radical axis of $\mathscr{C}_1$ and $\mathscr{C}_2$, we have $AB \cdot AX = AP \cdot AQ = AC \cdot AY$. From $AB = AC$, it follows that $AX = AY$, hence $XY \parallel BC$. But B, P, C are on the circle of diameter $[AO]$, so $ABPC$ is cyclic. But quadrilaterals $BXQP$ and $CYQP$ are also cyclic, therefore $\sphericalangle XQP = \sphericalangle ABP = 180° - \sphericalangle ACP = \sphericalangle YQP$, meaning that points X, Q, Y are collinear.

If E is the midpoint of $[XY]$, then AE is the perpendicular bisector of $[XY]$, and thus also the angle bisector of $\sphericalangle BAC$. Thus, $O \in AE$. But then OE is the parallel drawn through the midpoint of $[XY]$ to the bases of the right trapezoid $MXYN$, therefore OE passes through the midpoint of the side $[MN]$ as well. We conclude that $OM = ON$.



Soluţia 2: (user *77ant* pe AoPS)
The quadrilateral $MBPQ$ being cyclic, we have $\sphericalangle AQB = \sphericalangle PQB = \sphericalangle PMB = \sphericalangle OMB$ and $\sphericalangle MBO = \sphericalangle MPQ = 90° = \sphericalangle ABO$, hence $\sphericalangle MBO = 90° + \sphericalangle OBQ = \sphericalangle ABQ$. It follows that triangles ABQ and OBM are similar. In the same way, we get that $\Delta ACQ \sim \Delta NCO$, and thus we have $\frac{OM}{OB} = \frac{AQ}{AB} = \frac{AQ}{AC} = \frac{ON}{OC}$. As $OB = OC$, we get that $OM = ON$.



Other solutions can be found on AoPS.