2023 STARS OF MATHEMATICS, Junior Grade — SOLUTIONS

Problem 1. Determine all pairs (p, q) of primes for which $p^2 + 5pq + 4q^2$ is the square of an integer.

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Solution. The required pairs are $(5, 11)$, $(13, 3)$ and $(7, 5)$. A routine check shows that the corresponding squares are 28^2, 20^2 and 18^2, respectively.

Let $p^2 + 5pq + 4q^2 = n^2$, where n is a non-negative integer. Alternatively, but equivalently, $pq = n^2 - (p + 2q)^2 = (n - p - 2q)(n + p + 2q)$. The second factor on the right hand side is a divisor of pq, greater than both p and q. Since p and q are both prime, $n + p + 2q = pq$, so $n - p - 2q = 1$. Subtract the two to get $pq - 1 = (n + p + 2q) - (n - p - 2q) = 2p + 4q$. Alternatively, but equivalently, $(p - 4)(q - 2) = 9$. Clearly, $q \geq 2$, so $p \geq 5$. The factors are then 1 and 9 or 9 and 1 or 3 and 3, whence the required pairs.

Problem 2. A triangle is tiled with a finite number of triangles whose sides all have an odd length. Prove that the perimeter of the triangle is an integer of the same parity as the number of triangles in the tiling.

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Solution. Let Δ be the triangle under consideration. Since Δ is convex, the tiling triangles fall into two classes: Those having all edges inside Δ, and those having at least one edge on the boundary of Δ.

Every inner edge of a triangle is subdivided into one or more 'short' segments by (the boundaries of) some other triangles on the opposite side. Each short segment is shared by exactly two triangles. Notice further that every short segment lies along a unique segment of maximal length which is a concatenation of non-overlapping inner edges coming from the triangles on the same side of that segment. Hence, the total length of the short segments along one of maximal length is integer. Consequently, so is the total length s of all short segments.

Clearly, every outer edge (lying on the boundary of Δ) belongs to a single triangle, and the total length of all outer edges is the perimeter of Δ.

Finally, let t be the number of triangles, and let S be the sum of their perimeters. Since the sides of each triangle all have an odd length, t and S have like parities. By the preceding, the perimeter of Δ is $S - 2s$, and the conclusion follows.

Problem 3. Let ABC be an acute triangle with $AB < AC$, and let D be a variable point interior to the side AB. The parallel through D to BC crosses AC at E. The perpendicular bisector of DE crosses BC at F. The circles BDF and CEF cross again at K. Prove that the line FK passes through a fixed point.

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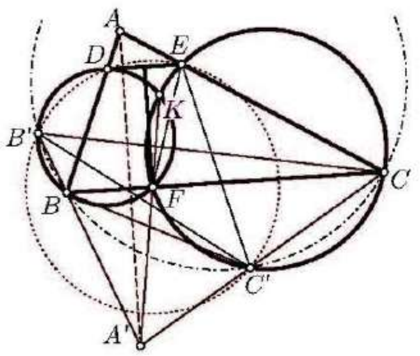
Solution. We will prove that FK passes through the reflection A' of A in the line BC which is clearly a fixed point. Alternatively, but equivalently, we are to show that A' has equal powers with respect to the circles γ_B through B, D, F, and γ_C through C, E, F.

Let $A'B$ cross γ_B again at B', and let $A'C$ cross γ_C again at C'. The argument hinges on two facts below:

(1) B', D, E and C' all lie on a circle γ centred at F; and

(2) The quadrangle $BB'CC'$ is cyclic.



Statement **(2)** clearly implies that A' has equal powers with respect to γ_B and γ_C, so we proceed to prove the two statements above.

To prove **(1)**, read angles from γ_B: $\angle B'DF = \angle A'BF = \angle FBD = \angle FB'D$, so $FB' = FD$. Similarly, $FC' = FE$, and since $FD = FE$, statement **(1)** follows.

To prove **(2)**, read angles from γ, γ_C and γ_B:

$$\begin{aligned}\angle B'C'C &= \angle B'C'E + \angle EC'C = (180^\circ - \angle B'DE) + \angle EFC \\ &= 180^\circ - (\angle B'DF + \angle FDE) + \angle FDE \qquad (\text{since } \angle EFC = \angle FED = \angle FDE) \\ &= 180^\circ - \angle B'DF = \angle B'BF = \angle B'BC.\end{aligned}$$

This establishes **(2)** and completes the proof.

Problem 4. Determine all integers $n \geq 3$ satisfying the following condition: There exist pairwise distinct real numbers $a_1, a_2, \ldots, a_n$ such that the $\frac{1}{2}n(n-1)$ sums $a_i + a_j$, $1 \leq i < j \leq n$, ordered increasingly, form an arithmetic sequence (the difference of every two consecutive sums is the same).

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Solution. The required integers are $n = 3$ and $n = 4$. In the former case, let $(a_1, a_2, a_3) = (1, 2, 3)$; the sums of pairs form the arithmetic sequence 3, 4, 5. In the other case, let $(a_1, a_2, a_3, a_4) = (1, 3, 4, 5)$; the sums of pairs form the arithmetic sequence 4, 5, 6, 7, 8, 9.

Since the n numbers are pairwise distinct, so are the $\frac{1}{2}n(n-1)$ sums of pairs — otherwise, the difference of the arithmetic sequence would be zero, so the first and the second smallest sums would be equal, hence the second and the third smallest numbers would be equal, and we would reach a contradiction.

Let now $n \geq 5$, and suppose, if possible, that $a_1, a_2, \ldots, a_n$ satisfy the condition in the statement. Without loss of generality, we may and will assume that $a_1 < a_2 < \cdots < a_n$. Let d be the difference of the corresponding arithmetic sequence of sums.

The smallest sum is $a_1 + a_2$, and the second smallest sum is $a_1 + a_3$, so $a_3 - a_2 = d$. The largest sum is $a_{n-1} + a_n$, and the second largest sum is $a_{n-2} + a_n$, so $a_{n-1} - a_{n-2} = d$. Consequently, $a_2 + a_{n-1} = (a_3 - d) + (a_{n-2} + d) = a_3 + a_{n-2}$. If $n \geq 6$, the leftmost sum and the rightmost sum correspond to distinct pairs, so they are at leat d distance apart. This contradiction forces $n = 5$.

Let $n = 5$. By the preceding, $a_3 - a_2 = d$, and $a_4 - a_3 = d$, so $2a_3 = a_2 + a_4$. To reach a contradiction, we present two approaches.

1st Approach. Evaluate the sum $s = \sum_{1 \leq i < j \leq 5}(a_i + a_j)$ in two different ways. On the one hand, $s = 5(a_1 + a_2 + a_4 + a_5)$. On the other hand, $s = 4(a_1 + a_2 + a_3 + a_4 + a_5)$, since each a_k

occurs in exactly four sums $a_i + a_j$, $i < j$. Equate the two and clear out like terms, to get $a_1 + a_2 + a_4 + a_5 = 4a_3 = 2(a_2 + a_4)$. Consequently, $a_1 + a_5 = a_2 + a_4$ which is the desired contradiction.

2nd Approach. Notice that the third smallest sum is $a_1 + a_4$ which is d larger than $a_1 + a_3$, and the the third largest sum is $a_2 + a_5$. Between these lie $a_1 + a_5$ and the consecutive sums $a_2 + a_3 < a_2 + a_4 < a_3 + a_4$. Then $a_1 + a_5$ is either the fourth smallest sum or the fourth largest.

Without loss of generality, we may and will assume that $a_1 + a_5$ is the fourth smallest sum. The string of ten sums is then

$$a_1+a_2 < a_1+a_3 < a_1+a_4 < a_1+a_5 < a_2+a_3 < a_2+a_4 < a_3+a_4 < a_2+a_5 < a_3+a_5 < a_4+a_5.$$

Evaluate $a_5 - a_4$ in two different ways. On the one hand, $a_5 - a_4 = (a_1+a_5) - (a_1+a_4) = d$, by the third inequality from the left. On the other hand,

$$a_5 - a_4 = \big((a_2 + a_5) - (a_3 + a_4)\big) + (a_3 - a_2) = d + d = 2d,$$

by the third inequality from the right and $a_3 - a_2 = d$. This is the desired contradiction.