Problema săptămânii 378

a) Numerele reale pozitive a, b, c satisfac $abc = 1$. Arătaţi că

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3.$$

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b) Numerele reale pozitive a, b, c, d satisfac $abcd = 1$. Arătaţi că

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq 4.$$

MEMO, 2023, proba pe echipe

Soluţie:

a) Din inegalitatea mediilor avem

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3\sqrt[3]{\frac{1+ab}{1+a} \cdot \frac{1+bc}{1+b} \cdot \frac{1+ca}{1+c}} =$$

$$3\sqrt[3]{abc \cdot \frac{1+ab}{1+a} \cdot \frac{1+bc}{1+b} \cdot \frac{1+ca}{1+c}} = 3\sqrt[3]{\frac{c(1+ab)}{1+a} \cdot \frac{a(1+bc)}{1+b} \cdot \frac{b(1+ca)}{1+c}} =$$

$$3\sqrt[3]{\frac{c+1}{1+a} \cdot \frac{a+1}{1+b} \cdot \frac{b+1}{1+c}} = 3,$$

cu egalitate dacă $\frac{1+ab}{1+a} = \frac{1+bc}{1+b} = \frac{1+ca}{1+c} = 1$, adică pentru $a = b = c = 1$.

b) Avem

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} =$$

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{ab(1+cd)}{ab(1+c)} + \frac{bc(1+da)}{bc(1+d)} =$$

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{ab+1}{ab+abc} + \frac{bc+1}{bc+bcd} =$$

$$(1+ab) \cdot \left(\frac{1}{1+a} + \frac{1}{ab+abc}\right) + (1+bc) \cdot \left(\frac{1}{1+b} + \frac{1}{bc+bcd}\right).$$

Din inegalitatea Cauchy-Buniakowsky-Schwarz (forma Titu Andreescu) avem

$$\frac{1}{1+a} + \frac{1}{ab+abc} \geq \frac{4}{1+a+ab+abc}$$

şi

$$\frac{1}{1+b} + \frac{1}{bc+bcd} \geq \frac{4}{1+b+bc+bcd}.$$

Atunci

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq \frac{4(1+ab)}{1+a+ab+abc} + \frac{4(1+bc)}{1+b+bc+bcd} =$$

$$\frac{4(1+ab)}{1+a+ab+abc}+\frac{4a(1+bc)}{a(1+b+bc+bcd)}=\frac{4(1+ab)}{1+a+ab+abc}+\frac{4(a+abc)}{a+ab+abc+1}=$$

$$\frac{4(1+ab+a+abc}{1+a+ab+abc}=4.$$

Egalitatea are loc atunci când $abcd=1$ şi avem egalitate în fiecare din cele două CBS-uri, adică $1+a=ab(1+c)$ şi $1+b=bc(1+d)$. Trebuie, deci, ca $a(1-bc)=ab-1$ şi $1-bc=b(cd-1)$, adică $ab-1=a(1-bc)=ab(cd-1)=1-ab$. Deducem că egalitatea are loc pentru $ab=bc=cd=da=1$, deci $a=c=A>0$ şi $b=d=\dfrac{1}{A}$.

Am primit soluţii de la: *Titu Zvonaru, Cristian Muth* (două soluţii), *Gheorghe Iurea, Marian Cucoaneş, Ioana Vlădoiu* şi *Eric-Dimitrie Cismaru.*

Problem of the week no. 378

a) Positive real numbers a,b,c satisfy $abc=1$. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ca}{1+c}\geq 3.$$

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b) Positive real numbers a,b,c,d satisfy $abcd=1$. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geq 4.$$

MEMO, 2023, team competition

Solution:

a) From the AM-GM inequality we have

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ca}{1+c}\geq 3\sqrt[3]{\frac{1+ab}{1+a}\cdot\frac{1+bc}{1+b}\cdot\frac{1+ca}{1+c}}=$$

$$3\sqrt[3]{abc\cdot\frac{1+ab}{1+a}\cdot\frac{1+bc}{1+b}\cdot\frac{1+ca}{1+c}}=3\sqrt[3]{\frac{c(1+ab)}{1+a}\cdot\frac{a(1+bc)}{1+b}\cdot\frac{b(1+ca)}{1+c}}=$$

$$3\sqrt[3]{\frac{c+1}{1+a}\cdot\frac{a+1}{1+b}\cdot\frac{b+1}{1+c}}=3,$$

with equality holding when $\dfrac{1+ab}{1+a}=\dfrac{1+bc}{1+b}=\dfrac{1+ca}{1+c}=1$, i.e. for $a=b=c=1$.

b) We have

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}=$$

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{ab(1+cd)}{ab(1+c)}+\frac{bc(1+da)}{bc(1+d)}=$$
$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{ab+1}{ab+abc}+\frac{bc+1}{bc+bcd}=$$
$$(1+ab)\cdot\left(\frac{1}{1+a}+\frac{1}{ab+abc}\right)+(1+bc)\cdot\left(\frac{1}{1+b}+\frac{1}{bc+bcd}\right).$$

From Titu's Lemma we have

$$\frac{1}{1+a}+\frac{1}{ab+abc}\geq\frac{4}{1+a+ab+abc}$$

şi

$$\frac{1}{1+b}+\frac{1}{bc+bcd}\geq\frac{4}{1+b+bc+bcd}.$$

Atunci

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geq\frac{4(1+ab)}{1+a+ab+abc}+\frac{4(1+bc)}{1+b+bc+bcd}=$$
$$\frac{4(1+ab)}{1+a+ab+abc}+\frac{4a(1+bc)}{a(1+b+bc+bcd)}=\frac{4(1+ab)}{1+a+ab+abc}+\frac{4(a+abc)}{a+ab+abc+1}=$$
$$\frac{4(1+ab+a+abc}{1+a+ab+abc}=4.$$

Equality holds when $abcd=1$ and we have equality both times we applied Titu's Lemma, i.e. $1+a=ab(1+c)$ şi $1+b=bc(1+d)$. We thus need $a(1-bc)=ab-1$ and $1-bc=b(cd-1)$, i.e. $ab-1=a(1-bc)=ab(cd-1)=1-ab$. It follows that equality holds when $ab=bc=cd=da=1$, i.e. $a=c=A>0$ and $b=d=\dfrac{1}{A}$.