Problema săptămânii 375

Dacă diferenţa cuburilor a două numere naturale consecutive este pătratul unui număr natural, atunci acel număr natural este suma pătratelor a două numere consecutive. (De exemplu, $8^3 - 7^3 = 13^2$, iar $13 = 2^2 + 3^2$.)

Robert Lyness, 1948

Soluţie: Dacă $(n+1)^3-n^3 = m^2$, adică $3n^2+3n+1 = m^2$, atunci $4m^2 = 3(2n+1)^2+1$, deci $3(2n+1)^2 = (2m-1)(2m+1)$. Să observăm că $(2m-1, 2m+1) = 1$, deci trebuie ca fie $2m-1 = k^2$, $2m+1 = 3j^2$, cu $k, j \in \mathbb{N}$, $kj = 2n-1$, fie $2m-1 = 3k^2$, $2m+1 = j^2$, cu $k, j \in \mathbb{N}$, $kj = 2n+1$. Dar al doilea caz conduce la $j^2 = 3k^2 + 2 \equiv 2 \pmod 3$, ceea ce nu este posibil. Deoarece k este impar, $k = 2u + 1$, deci $2m - 1 = 4u^2 + 4u + 1$, adică $m = 2u^2 + 2u + 1 = u^2 + (u + 1)^2$.

Am primit soluţii de la: *Cristian Vergelea, Cristian Muth, Eric-Dimitrie Cismaru, Gheorghe Iurea, Titu Zvonaru, Ioana Vlădoiu* şi *David Mathe.*

Problem of the week no. 375

Show that if the difference of the cubes of two consecutive integers is the square of an integer, then this integer is the sum of the squares of two consecutive integers. (For example, $8^3 - 7^3 = 13^2$, and $13 = 2^2 + 3^2$.)

Robert Lyness, 1948

Solution: If $(n + 1)^3 - n^3 = m^2$, i.e. $3n^2 + 3n + 1 = m^2$, then $4m^2 = 3(2n + 1)^2 + 1$, hence $3(2n + 1)^2 = (2m - 1)(2m + 1)$. As $(2m - 1, 2m + 1) = 1$, we must have either $2m - 1 = k^2$, $2m + 1 = 3j^2$, with $k, j \in \mathbb{N}$, $kj = 2n - 1$, or $2m - 1 = 3k^2$, $2m + 1 = j^2$, with $k, j \in \mathbb{N}$, $kj = 2n + 1$. But the second case leads to $j^2 = 3k^2 + 2 \equiv 2 \pmod 3$, which is not possible. Note that k is odd, hence, $k = 2u + 1$ with some integer u, hence $2m - 1 = 4u^2 + 4u + 1$, i.e. $m = 2u^2 + 2u + 1 = u^2 + (u + 1)^2$.