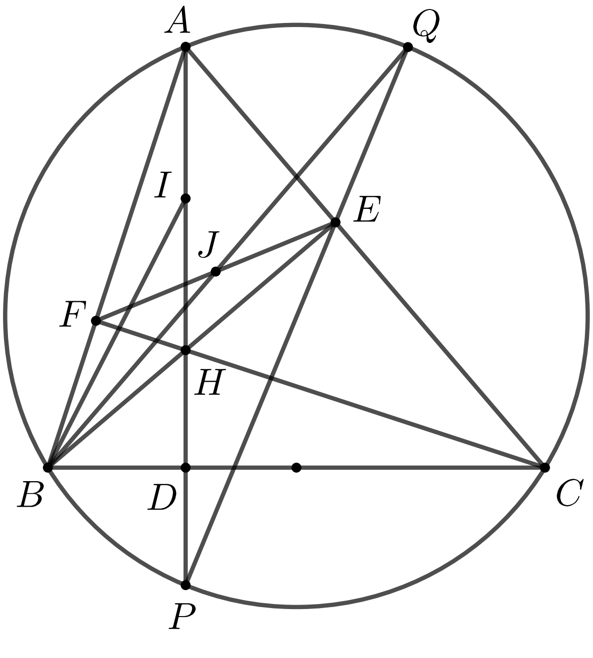
Problema săptămânii 361

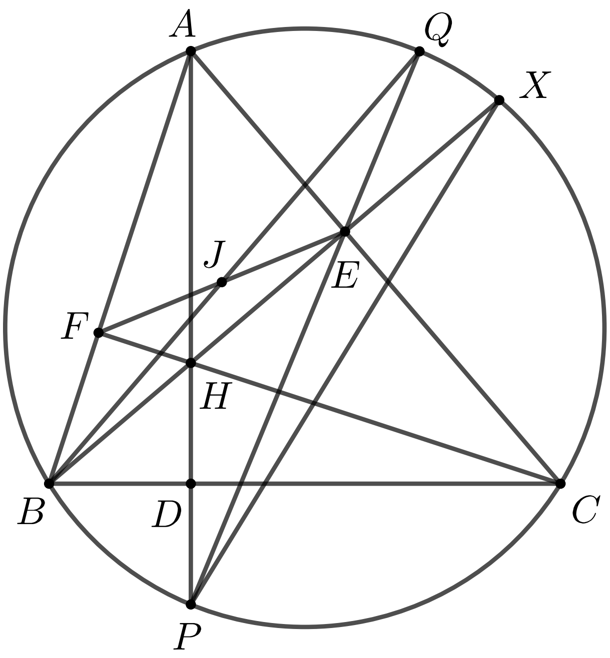
Fie ABC un triunghi ascuţitunghic înscris în cercul (O), iar AD, BE, CF înălţimile sale. Dreapta AD intersectează din nou cercul (O) în P, iar dreapta PE intersectează din nou cercul (O) în Q. Demonstraţi că BQ trece prin mijlocul segmentului EF.

Dang Nguyen, Romantics of Geometry, 10970

Soluţia 1: (*Nguyen Duc Hieu*, Romantics of Geometry)
Fie I, J mijloacele segmentelor AH, respectiv EF. Vom arăta că B, J, Q sunt coliniare.
Patrulaterul $AFHE$ fiind inscriptibil, triunghiurile BHA şi BFE sunt asemenea (UU), deci şi jumătăţile lor, triunghiurile BIA şi BJE, sunt asemenea (LUL).
Deducem că $\sphericalangle JBE = \sphericalangle ABI$. (1)
Se ştie că $HD = DP$. Atunci $HI \cdot HP = HI \cdot 2HD = 2HI \cdot HD = AH \cdot HD = HB \cdot HE$ (din puterea punctului H faţă de cercul de diametru AB, circumscris patrulaterului $ABDE$). Deducem de aici că patrulaterul $BPEI$ este inscriptibil, deci $\sphericalangle BIP = \sphericalangle BEP$, de unde $\sphericalangle AIB = \sphericalangle QEB$. Avem şi $\sphericalangle BAI = \sphericalangle BQE$ (patrulaterul $ABPQ$ este inscriptibil), deci triunghiurile ABI şi QBE sunt asemenea.
Aşadar, $\sphericalangle ABI = \sphericalangle QBE$. (2).
Din (1) şi (2) rezultă că punctele B, J, Q sunt coliniare, de unde concluzia.



Soluţia 2: (*Ovidiu Lazăr*, Romantics of Geometry)
Fie X al doilea punct de intersecţie dintre BE şi cercul circumscris triunghiului ABC, iar J intersecţia dintre BQ şi EF. Din $AFHE$ inscriptibil rezultă asemănarea triunghiurilor BFE şi PHX, respectiv BFJ şi PHE. Dar, cum E este mijlocul lui $[HX]$, rezultă că J este mijlocul lui $[EF]$.

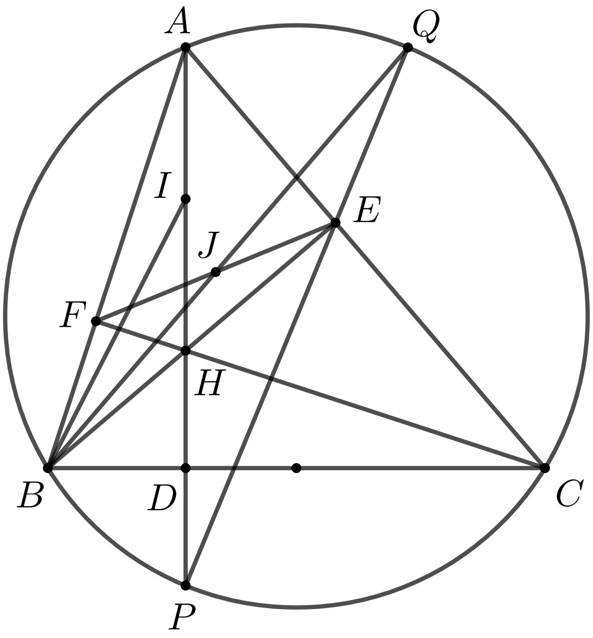


Problem of the week no. 361

Let ABC be an acute triangle inscribed in circle (O) with altitudes AD, BE, CF. Line AD intersects the circle (O) again at P, and line PE intersects the circle (O) again at Q. Prove that BQ passes through the midpoint of EF.

Dang Nguyen, Romantics of Geometry, 10970

Solution 1: (*Nguyen Duc Hieu*, Romantics of Geometry)
Let I, J be the midpoints of AH and EF, respectively. We have to prove that points B, J, Q are collinear. Quadrilateral $AFHE$ is cyclic, therefore triangles BHA and BFE are similar (AA), hence so are their halves: triangles BIA and BJE are also similar (SAS).
It follows that $\sphericalangle JBE = \sphericalangle ABI$. (1)
It is well known that $HD = DP$. Then $HI \cdot HP = HI \cdot 2HD = 2HI \cdot HD = AH \cdot HD = HB \cdot HE$ (from the power of point H with respect to the circle of diameter AB, the circumcircle of quadrilateral $ABDE$). It follows from here that the quadrilateral $BPEI$ is cyclic, therefore $\sphericalangle BIP = \sphericalangle BEP$, and hence $\sphericalangle AIB = \sphericalangle QEB$. We also have $\sphericalangle BAI = \sphericalangle BQE$ ($ABPQ$ is cyclic), hence triangles ABI and QBE are similar.
Thus, $\sphericalangle ABI = \sphericalangle QBE$. (2).
From (1) and (2) it follows that points B, J, Q are collinear, and the conclusion.



Solution 2: (*Ovidiu Lazăr*, Romantics of Geometry)
Let X the second intersection point between BE and the circumcircle of triangle ABC, and let J be the intersection point between lines BQ and EF. As $AFHE$is cyclic, we obtain the similarity of triangles BFE and PHX, and also of triangles BFJ and PHE. But, as E is the midpoint of $[HX]$, it follows that J is the midpoint of $[EF]$.