Problem 1. If a, b, c are positive real numbers, prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geqslant a+b+c\,.$$

Problem 2. Let ABC be an equilateral triangle and let E be a point on the line segment BC. Let ℓ be the line through A parallel to BC and let K be the point on ℓ such that KE is perpendicular to BC. The circle with centre K and radius KE intersects the sides AB and AC at M and N respectively. The perpendicular on AB at M intersects ℓ at D, and the perpendicular on AC at N intersects ℓ at F.

Show that the point of intersection of the angle bisectors of angles $\angle MDA$ and $\angle NFA$ belongs on the line KE. **(Source: JBMO Shortlist 2022-G4)**

Problem 3. Anna and Bob, with Anna starting first, alternately colour the integers of the set $S = \{1, 2, \ldots, 2022\}$ red or blue. At their turn, each one can colour any uncoloured number of S they wish, with any colour they wish. The game ends when all numbers of S get coloured.

Let N be the number of pairs $(a, b) \in S^2$ where a, b have the same colour, and $b–a = 3$.

Anna wishes to maximize N. What is the maximum value of N that she can achieve regardless of how Bob plays? **(Source: JBMO Shortlist 2022-C1)**

Problem 4. Consider the sequence $u_0, u_1, u_2, \ldots$ defined by $u_0 = 0, u_1 = 1$ and

$$u_n = 6u_{n-1} + 7u_{n-2}$$

for $n \geqslant 2$. Show that there are no non-negative integers a, b, c, n such that

$$ab(a+b)(a^2+ab+b^2) = c^{2022} + 42 = u_n\ .$$

(Source: JBMO Shortlist 2022-N4)