Problem 1 - Solution 1. We have

$$\frac{a^2+b^2}{a+b}-\frac{a+b}{2}=\frac{2(a^2+b^2)-(a+b)^2}{2(a+b)}=\frac{a^2+b^2-2ab}{2(a+b)}=\frac{(a-b)^2}{2(a+b)}\geqslant 0\,.$$

Therefore

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geqslant\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}=a+b+c\,.$$

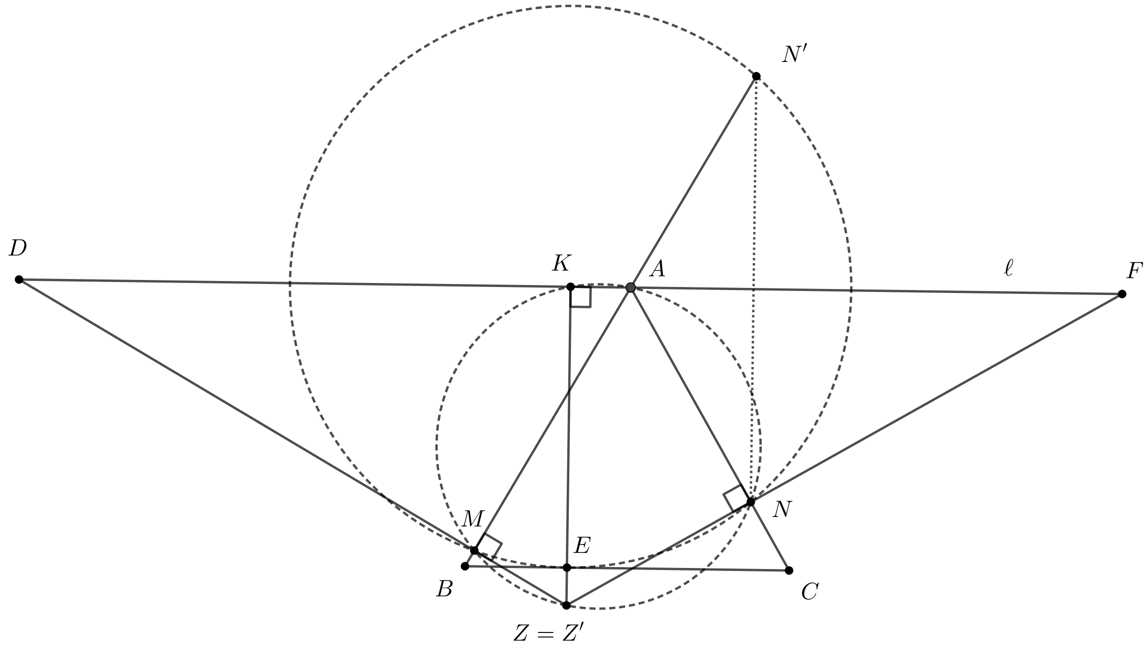
Problem 1 - Solution 2. From Cauchy-Schwarz we have

$$(1^2+1^2)(a^2+b^2)\geqslant(a+b)^2 \implies \frac{a^2+b^2}{a+b}\geqslant\frac{a+b}{2}\,.$$

We can now proceed as in Solution 1.

Problem 2 - Solution 1. Let N' be the symmetric point of N with respect to ℓ. Since $\angle DAB=\angle BAC=\angle CAF=60^\circ$, then N' belongs on AB. Since $KN=KN'$, then N' belongs on the circle with centre K and radius KE. Furthermore NAN' is isosceles with $\angle NAN'=120^\circ$ and $\angle ANN'=\angle AN'N=30^\circ$. Therefore $\angle MKN=2\angle MN'N=60^\circ$. Thus A,K,M,N are concyclic, say on the circle ω.

Let Z be the point of intersection of DM and KE. Since $\angle ZMA=90^\circ=\angle ZKA$, then $AKZM$ is cyclic and so Z belongs on ω. Similarly, if Z' is the point of intersection of FN with KE, then Z' also belongs on ω. But KE can have only one other point of intersection with ω apart from K, therefore $Z=Z'$.



We have $\angle ZDF = \angle MDA = 30^\circ = \angle NFA = \angle ZFD$. So the triangle DZF is isosceles. Since ZK is the altitude of the triangle DZF, it is also a bisector. So the incenter of the triangle DZF belongs on EK and therefore the required conclusion follows.

Problem 2 - Solution 2. We start by showing that A, K, M, N are concyclic, say on the circle ω.

We may assume that E is not the midpoint of BC as otherwise $A = K$ and the required conclusion is immediate. Without loss of generality we have $EB < EC$. From sine law in the triangles KAM and KAN we have

$$\frac{KA}{\sin(\angle KMA)} = \frac{KM}{\sin(60^\circ)} = \frac{KN}{\sin(60^\circ)} = \frac{KA}{\sin(\angle KNA)}.$$

Since the angles $\angle KMA$ and $\angle KNA$ are acute (the angles $\angle KAN$ and AKM are obtuse) then $\angle KMA = \angle KNA$. So A, K, M, N are concyclic.

Let A' be the antipodal point of A with respect to ω. Then $\angle A'MA = \angle A'NA = 90^\circ$ and so A' is the point of intersection of DM and FN. Furthermore $\angle A'KA = 90^\circ$, so A' belongs on EK.

Since $KM = KN$ and $\angle MKN = \angle MAN = 60^\circ$, then KMN is equilateral. Then $\angle MA'K = \angle MNK = 60^\circ = \angle NMK = \angle NA'K$. So $A'K$ bisects $\angle DA'F$. Therefore the incenter of triangle $DA'F$ belongs on EK and the required conclusion follows.

Problem 3. We show that Anna can achieve $N = 1008$ but not a bigger value.

Let $A = \{1, 4, 7, \dots, 2020\}, B = \{2, 5, 8, \dots, 2021\}$ and $C = \{3, 6, 9, \dots, 2022\}$.

In each of her moves, Anna tries to find two numbers at distance 3 with one of them being coloured and the other uncoloured. If she can achieve that, then she colours the uncoloured number with the same colour as that of the coloured one.

Anna fails in the above task if and only if in each of the sets A, B, C either all numbers are coloured or all numbers are uncoloured. This can occur at most three times.

So Anna can succeed in her above task at least $1011-3$ times. Each time that she succeeds, she increases the value of N by 1. So she can definitely achieve $N \geqslant 1008$.

Let M be the number of pairs (a, b) with $a, b \in S$ having a different colour and $b - a = 3$. At the end of the game we will have $M + N = 2019$ since each of the pairs $(1,4), (2,5), \ldots, (2019, 2022)$ contributes 1 to either M or N.

Bob considers that 2011 pairs

$$(1,4),(2,5),(3,6),(7,10),(8,11),(9,12),\ldots,(2017,2020),(2018,2021),(2019,2022),$$

which are pairwise disjoint. If Anna colours the number x, then Bob gives a different colour to the number that belongs to the same pair as x.

With this strategy Bob can achieve $M \geqslant 1$ which gives $N \leqslant 2019 - 1011 = 1008$.

Problem 4. We first observe that

$$\begin{aligned}(a+b)^5 - a^5 - b^5 &= 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 \\ &= 5ab(a^3 + 2a^2b + 2ab^2 + b^3) \\ &= 5ab(a+b)(a^2+ab+b^2).\end{aligned}$$

Since $x^5 \equiv \{-1,0,1\} \bmod 11$, then $(a+b)^5 - a^5 - b^5 \equiv 0, \pm1, \pm2, \pm3 \bmod 11$, therefore

$$ab(a+b)(a^2+ab+b^2) \equiv -2\left((a+b)^5 - a^5 - b^5\right) \equiv 0, \pm2, \pm4, \pm6 \bmod 11. \tag{1}$$

From Fermat's Little Theorem, if $11 \nmid c$, then $c^{2022} = (c^{10})^{202} \cdot c^2 \equiv c^2 \equiv 1,3,4,5,9 \bmod 11$. So for every integer c we have $c^{2022} \equiv 0,1,3,4,5,9 \bmod 11$. Since in addition $42 \equiv -2 \bmod 11$, then

$$c^{2022} + 42 \equiv 1,2,3,7,9,10 \bmod 11. \tag{2}$$

Finally, the first terms of the sequence u_n modulo 11 are $0,1,6,-1,3,0,-1,-6,1,-3,0,1$. Therefore the sequence is periodic modulo 11 and

$$u_n \equiv \{0, \pm1, \pm3, \pm5\} \bmod 11. \tag{3}$$

Since (1), (2), (3) cannot all be satisfied concurrently, then the equation has no solutions in non-negative integers.