Problem 1. We have

$$\begin{aligned} x^2 - y^2 = 5x &\iff x^2 - 5x - y^2 = 0 \\ &\iff \left(x - \frac{5}{2}\right)^2 - y^2 = \frac{25}{4} \\ &\iff (2x-5)^2 - 4y^2 = 25 \\ &\iff (2x - 5 - 2y)(2x - 5 + 2y) = 25\,. \end{aligned}$$

Writing $a = 2x - 5 - 2y$ and $b = 2x - 5 + 2y$ we must have

$$(a, b) \in \{(1, 25), (5, 5), (25, 1), (-1, -25), (-5, -5), (-25, -1)\}\,.$$

Since $a + b = 2(2x - 5)$, then $x = (a + b + 10)/4$. Since $b - a = 4y$, then $y = (b - a)/4$. So we end up with the solutions

$$(x, y) \in \{(9, 6), (5, 0), (9, -6), (-4, -6), (0, 0), (-4, 6)\}\,. \quad \square$$

Problem 2. We consider the following cases:

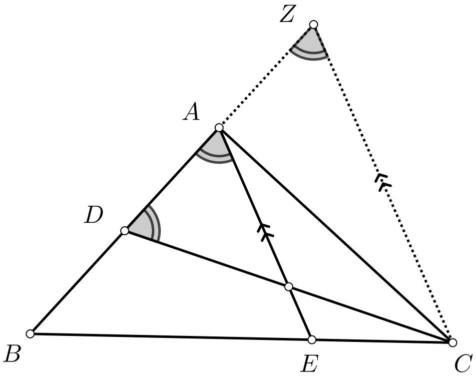
- If $p = 2$, then $2p^2 + 5^{p-1} = 8 + 5 = 13$ is prime.
- If $p = 3$, then $2p^2 + 5^{p-1} = 18 + 25 = 43$ is prime.

- If $p > 3$, then $3 \nmid p^2$ and so $p^2 \equiv 1 \bmod 3$. Furthermore, p is odd, so $p = 2k+1$, therefore $5^{p-1} = 5^{2k} = 25^k \equiv 1 \bmod 3$. Thus

$$2p^2 + 5^{p-1} \equiv 2 \cdot 1 + 1 \equiv 0 \bmod 3\,.$$

So $2p^2 + 5^{p-1}$ is not prime. □

Problem 3 - Solution 1.



Let (ε) be the parallel through C to AE and let Z be the point of intersection of (ε) with BA. We have

$$\angle AZC = \angle BAE = \angle CDA$$

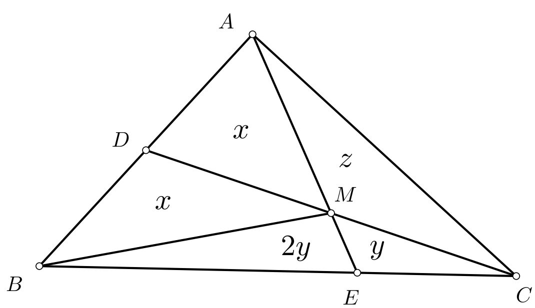
so the triangle DCZ is isosceles with $CD = CZ$.

From Thales Theorem on the triangle BCZ we have

$$2 = \frac{BE}{EC} = \frac{BA}{AZ} = \frac{2AD}{AZ} \implies AZ = AD\,.$$

So CA is a median of the isosceles triangle DCZ and therefore it's also an altitude. Thus $\angle BAC = 90°$. □

Problem 3 - Solution 2.



Let M be the point of intersection of AE and CD. If $(MAD) = x$, then we must also have $(MBD) = x$. If $(MCE) = y$, then we must also have $(MBE) = 2y$. Assume also that $(MAC) = z$.

Since $(ABE) = 2(ACE)$ then $2x + 2y = 2(z+y)$ and therefore $x = z$. So $(MAC) = (MAD)$ which implies that $MC = MD$. Since the triangle MAD is isosceles, we also have $MD = MA$.

Thus the circle with centre M and radius $MA = MC = MD$ has diameter CD and passes through A. It follows that $\angle BAC = 90°$. □

Problem 4.

(a) Alice wins. Up to the choice of a_{2019} by Bob, Alice plays arbitrarily. Afterwards, she chooses a_{2020} and a_{2021} so that

$$a_1 - a_2 + \cdots - a_{2020} + a_{2021} \equiv 0 \bmod 11\,.$$

She can definitely achieve that as follows: It is enough for every $\ell \in \{0, 1, 2, \ldots, 10\}$ to be able to choose a_{2020} and a_{2021} such that $a_{2020} - a_{2021} \equiv \ell \bmod 11$.

For $\ell = 0, 1, \dots, 9$ she can choose $a_{2020} = \ell$ and $a_{2021} = 0$. For $\ell = 10$ she can choose $a_{2020} = 0$ and $a_{2021} = 1$.

Afterwards, whatever the choice of Bob for a_{2022}, Alice chooses $a_{2023} = a_{2022}$. We then have

$$a_1 - a_2 + \cdots - a_{2022} + a_{2023} \equiv 0 \bmod 11$$

and so the number is a multiple of 11.

(b) Bob wins. Up to the choice of a_{2021} by Alice, Bob plays arbitrarily. Afterwards, he chooses a_{2022} so that

$$a_1 + a_2 + \cdots + a_{2022} \equiv 2 \bmod 3\,.$$

He can definitely achieve that as follows: Let $S = a_1 + \cdots + a_{2021}$. If $S \equiv 0 \bmod 3$ he chooses $a_{2022} = 2$, if $S \equiv 1 \bmod 3$ he chooses $a_{2022} = 1$, and if $S \equiv 2 \bmod 3$ he chooses $a_{2022} = 0$.

If Alice wants to make the number a multiple of 5, she is forced to choose $a_{2023} = 0$ or $a_{2023} = 5$. If she chooses $a_{2023} = 0$ we have

$$a_1 + a_2 + \cdots + a_{2023} \equiv 2 \bmod 3\,,$$

while if she chooses $a_{2023} = 5$ we have

$$a_1 + a_2 + \cdots + a_{2023} \equiv 1 \bmod 3\,.$$

In both cases the number is not a multiple of 3 and therefore Bob wins.