Problem 1. We have

$$3 \cdot 2^k = n^2 - 1 = (n-1)(n+1)\,.$$

The left-hand-side is an even number. In order for the right-hand-side to be an even number n,must be odd. Then $n-1$ and $n+1$ are both even. Let d be the greatest common divisor of $n-1$ and $n+1$. Then d divides $n-1$ and $n+1$ so it also divides their difference $(n+1)-(n-1)=2$. Since $n+1$ and $n-1$ are both even, then we must have $d=2$.

So one of $n-1, n+1$ is a multiple of 2 but not of 4. Since it divides $3 \cdot 2^k$ it should therefore be equal to 2 or 6.

- If $n-1=2$, then $n=3$ and $3 \cdot 2^k = 8$, contradiction.
- If $n+1=2$, then $n=1$ and $3 \cdot 2^k = 0$, contradiction.
- If $n-1=6$, then $n=7$ and $3 \cdot 2^k = 48$ which gives $k=4$.
- If $n+1=6$, then $n=5$ and $3 \cdot 2^k = 24$ which gives $k=3$. □

Problem 2. We have

$$
\begin{aligned}
& \frac{a^2-a-c}{b}+\frac{b^2-b-c}{a}=a+b+2 \\
\Longrightarrow\ & \left(\frac{a^2-a-c}{b}-1\right)+\left(\frac{b^2-b-c}{a}-1\right)=a+b \\
\Longrightarrow\ & \frac{a^2-a-c-b}{b}+\frac{b^2-b-c-a}{a}=a+b \\
\Longrightarrow\ & \frac{a^2-(a+b+c)}{b}+\frac{b^2-(a+b+c)}{a}=a+b \\
\Longrightarrow\ & a^3+b^3-(a+b)(a+b+c)=ab(a+b) \\
\Longrightarrow\ & a^3+b^3=(a+b)(ab+a+b+c)\,.
\end{aligned}
$$

But

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

and since $a+b\neq 0$ we get

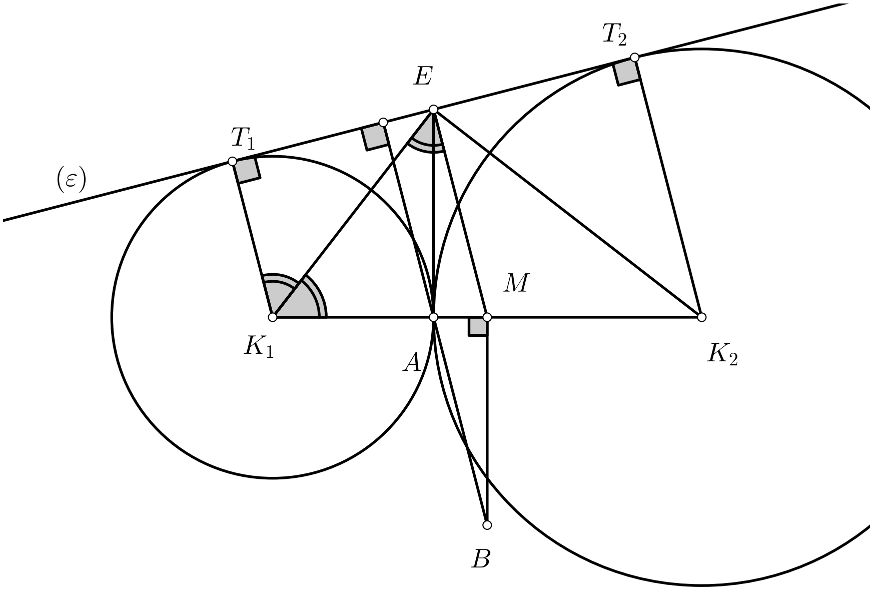
$$ab+a+b+c=a^2-ab+b^2\,.$$

Therefore

$$a+b+c=a^2-2ab+b^2=(a-b)^2$$

which is a perfect square as required. □

Problem 3. Let T_1 and T_2 be the points of intersection of the tangent (ε) with the circles (K_1, R_1) and (K_2, R_2) respectively. We draw the common tangent at A and let E be the point of intersection with (ε).



The triangles K_1T_1E and K_1AE are equal as they are right-angled with KE as a common hypotenuse. The same holds for the triangles K_2T_2E and K_2AE. Therefore

$$\angle T_1EK_1=\angle K_1EA \quad \text{and} \quad \angle T_2EK_2=\angle K_2EA\,.$$

Then

$$\begin{aligned} 180^\circ &= \angle T_1EK_1 + \angle K_1EA + \angle T_2EK_2 + \angle K_2EA \\ &= 2(\angle K_1EA + \angle K_2EA) = 2(\angle K_1EK_2)\,. \end{aligned}$$

So the triangle K_1EK_2 is right-angled. If M is the midpoint of K_1K_2 then EM is the median of the right-angled triangle, therefore $K_1K_2 = 2EM$.

It is enough to prove that $EABM$ is a parallelogram since then we will have $K_1K_2 = 2EM = 2AB$.

Since EA and BM are perpendicular to K_1K_2, then $EA \parallel BM$. Since $ME = MK_1$, then $\angle MEK_1 = \angle MK_1E = \angle AK_1E$. Since the triangles K_1T_1E and K_1AE are equal, then $\angle AK_1E = \angle T_1K_1E$. So $\angle MEK_1 = \angle T_1K_1E$ and therefore $ME \parallel K_1T_1$. We also have that $AB \parallel K_1T_1$ since both lines are perpendicular to (ε).

Since $AB \parallel ME$ and since also $EA \parallel BM$, then $EABM$ is a parallelogram as we wanted to show. □

Problem 4. Let $n_1 = a_1 + b_1, \ldots, n_{2023} = a_{2023} + b_{2023}$ be the sums that are written on the board. We have

$$\begin{aligned} n_1 + n_2 + \cdots + n_{2023} &= (a_1 + b_1) + \cdots + (a_{2023} + b_{2023}) \\ &= (a_1 + \cdots + a_{2023}) + (b_1 + \cdots + b_{2023}) \\ &= (1 + 2 + \cdots + 2023) + (-1 - 2 - \cdots - 2023) = 0 \end{aligned}$$

The number $n_1 + n_2 + \cdots + n_{2023}$ is even, therefore at least one out of $n_1, \ldots, n_{2023}$ must be even. Indeed otherwise, $n_1, \ldots, n_{2023}$ would all be odd. But then $n_1 + n_2 + \cdots + n_{2023}$ would also be odd, a contradiction.

So $2023 - 2^n = n_1 n_2 \cdots n_{2023}$ must be even. Then 2^n must be odd, which happens only when $n = 0$.

The value of $n = 0$ is possible. E.g. if from box A we pick the numbers $1, 2, \ldots, 2023$ in this order, and from box B. we pick the numbers $-2, -3, \ldots, -2023, -1$ in this order, then $n_1 = \cdots = n_{2022} = -1, n_{2023} = 2022$ and therefore $n_1 \cdots n_{2023} = 2022 = 2023 - 2^0$. □