Problema săptămânii 346

Fie $a, b, c > 0$ astfel încât $abc = 1$. Demonstraţi că

$$(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) \geq (a + bc + 1)(b + ca + 1)(c + ab + 1).$$

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Soluţia 1: Vom aplica inegalitatea lui Hölder în următoarea formă:

$$(x_1 + y_1 + z_1)(x_2 + y_2 + z_2)(x_3 + y_3 + z_3) \geq \left(\sqrt[3]{x_1x_2x_3} + \sqrt[3]{y_1y_2y_3} + \sqrt[3]{z_1z_2z_3}\right)^3,$$

oricare ar fi numerele reale pozitive x_i, y_i, z_i, $i = 1, 2, 3$.
Astfel,

$$(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) = (a^3 + abc + a)(b^3 + abc + b)(c^3 + abc + c) =$$

$$= (a^3 + 1 + a)(1 + b^3 + b)(1 + c^3 + c) \geq (a + bc + 1)^3.$$

Analog, avem relaţiile $(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) \geq (b + ca + 1)^3$ şi $(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) \geq (c + ab + 1)^3$.
Înmulţind aceste trei inegalităţi şi extrăgând radical de ordin 3 obţinem inegalitatea dorită.
Egalitatea are loc dacă şi numai dacă $a = b = c = 1$.

Soluția 2: (*Titu Zvonaru*)
După desfacerea parantezelor, inegalitatea se scrie

$$\Sigma a^3b^3 + \Sigma a^3b + \Sigma ab^3 + \Sigma a^3 \geq \Sigma a^2b + \Sigma ab^2 + \Sigma ab + \Sigma a. \qquad (1)$$

Inegalitatea (1) rezultă prin adunarea următoarelor relații, obținute cu inegalitatea mediilor şi inegalitatea lui Schur (și ținând cont că $abc = 1$) :

$$a^3b + a^3b + bc^3 \geq 3a,\ b^3c + b^3c + a^3c \geq 3b,\ ac^3 + ac^3 + ab^3 \geq 3c$$
$$a^3b^3 + a^3b^3 + c^3 \geq 3ab,\ b^3c^3 + b^3c^3 + a^3 \geq 3bc,\ a^3c^3 + a^3c^3 + b^3 \geq 3ac$$
$$2\left(a^3 + b^3 + c^3 + 3abc\right) \geq 2\left(a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2\right)$$
$$a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 \geq a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2$$
$$a^3b + b^3c + ac^3 \geq 3a^2b^2c^2,\ 2\left(ab^3 + bc^3 + a^3c\right) \geq 6abc.$$

Inegalitatea (1) se poate demonstra și folosind inegalitatea lui Muirhead. Pentru aceasta, după omogenizare, relația (1) se scrie sub forma

$$\sum_{\text{sim}} a^{\frac{11}{3}}b^{\frac{5}{3}}c^{\frac{2}{3}} + \sum_{cic} a^{\frac{10}{3}}b^{\frac{4}{3}}c^{\frac{4}{3}} + \sum_{cic} a^3b^3 \geq \sum_{\text{sim}} a^3b^2c + \sum_{cic} a^{\frac{8}{3}}b^{\frac{5}{3}}c^{\frac{5}{3}} + \sum_{cic} a^{\frac{7}{3}}b^{\frac{7}{3}}c^{\frac{4}{3}}$$

Această inegalitate este adevărată deoarece $\left(\frac{11}{3}, \frac{5}{3}, \frac{2}{3}\right)$ majorează $(3, 2, 1)$, $\left(\frac{10}{3}, \frac{4}{3}, \frac{4}{3}\right)$ pe $\left(\frac{8}{3}, \frac{5}{3}, \frac{5}{3}\right)$ și $(3, 3, 0)$ pe $\left(\frac{7}{3}, \frac{7}{3}, \frac{4}{3}\right)$.

Am primit soluţii de la: *Titu Zvonaru, Marian Cucoaneş, David Tache* şi *Ioan Viorel Codreanu.*

Problem of the week no. 346

Let $a, b, c > 0$ such that $abc = 1$. Prove that

$$(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) \geq (a + bc + 1)(b + ca + 1)(c + ab + 1).$$

AoPS user *squing*

Solution 1:

We apply Hölder's inequality in the following form:

$$(x_1 + y_1 + z_1)(x_2 + y_2 + z_2)(x_3 + y_3 + z_3) \geq \left(\sqrt[3]{x_1x_2x_3} + \sqrt[3]{y_1y_2y_3} + \sqrt[3]{z_1z_2z_3}\right)^3,$$

for all positive real numbers x_i, y_i, z_i, $i = 1, 2, 3$.

Thus,

$$(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) = (a^3 + abc + a)(b^3 + abc + b)(c^3 + abc + c) =$$
$$= (a^3 + 1 + a)(1 + b^3 + b)(1 + c^3 + c) \geq (a + bc + 1)^3.$$

Similarly we have $(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) \geq (b + ca + 1)^3$ and $(a^2 + bc + 1)(b^2 + ca + 1)(c^2 + ab + 1) \geq (c + ab + 1)^3$.

Multiplying these three inequalities gives the desired inequality lifted to the power 3. Equality holds if and only if $a = b = c = 1$.

Solution 2: (*Titu Zvonaru*)

Expanding, we obtain

$$\mathbf{\Sigma}a^3b^3 + \mathbf{\Sigma}a^3b + \mathbf{\Sigma}ab^3 + \mathbf{\Sigma}a^3 \geq \mathbf{\Sigma}a^2b + \mathbf{\Sigma}ab^2 + \mathbf{\Sigma}ab + \mathbf{\Sigma}a. \qquad (1)$$

The inequality (1) can be obtained by adding the following inequalities that are consequences of the AM$-$GM inequality and Schur's inequality (also using $abc = 1$) :

$$a^3b + a^3b + bc^3 \geq 3a,\ b^3c + b^3c + a^3c \geq 3b,\ ac^3 + ac^3 + ab^3 \geq 3c$$
$$a^3b^3 + a^3b^3 + c^3 \geq 3ab,\ b^3c^3 + b^3c^3 + a^3 \geq 3bc,\ a^3c^3 + a^3c^3 + b^3 \geq 3ac$$
$$2\left(a^3 + b^3 + c^3 + 3abc\right) \geq 2\left(a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2\right)$$
$$a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 \geq a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2$$
$$a^3b + b^3c + ac^3 \geq 3a^2b^2c^2,\ 2\left(ab^3 + bc^3 + a^3c\right) \geq 6abc.$$

The inequality (1) can be proven also by using Muirhead's inequality. First, one needs to make (1) homogeneous:

$$\sum_{\text{sim}} a^{\frac{11}{3}}b^{\frac{5}{3}}c^{\frac{2}{3}} + \sum_{cic} a^{\frac{10}{3}}b^{\frac{4}{3}}c^{\frac{4}{3}} + \sum_{cic} a^3b^3 \geq \sum_{\text{sim}} a^3b^2c + \sum_{cic} a^{\frac{8}{3}}b^{\frac{5}{3}}c^{\frac{5}{3}} + \sum_{cic} a^{\frac{7}{3}}b^{\frac{7}{3}}c^{\frac{4}{3}}$$

This inequality is true because $\left(\frac{11}{3}, \frac{5}{3}, \frac{2}{3}\right)$ majorizes $(3, 2, 1)$, $\left(\frac{10}{3}, \frac{4}{3}, \frac{4}{3}\right)$ majorizes $\left(\frac{8}{3}, \frac{5}{3}, \frac{5}{3}\right)$ and $(3, 3, 0)$ majorizes $\left(\frac{7}{3}, \frac{7}{3}, \frac{4}{3}\right)$.