Problema săptămânii 338

Fie a, b, c numere reale pozitive. Demonstraţi inegalitatea

$$\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+a)(b+c)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}}\leq 1.$$

Walther Janous, Crux Mathematicorum, iunie 1991

Soluţia 1: Avem $\sqrt{(x+y)(x+z)} \geq \sqrt{xy}+\sqrt{xz}$, $\forall\, x, y, z > 0$.
Într-adevăr, prin ridicare la pătrat, inegalitatea precedentă revine la $x^2+yz \geq 2x\sqrt{yz}$, adică $(x-\sqrt{yz})^2 \geq 0$, evident adevărată.
Folosind această inegalitate, avem

$$\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+a)(b+c)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}}\leq$$

$$\frac{a}{a+\sqrt{ab}+\sqrt{ac}}+\frac{b}{b+\sqrt{ba}+\sqrt{bc}}+\frac{c}{c+\sqrt{ca}+\sqrt{cb}}=$$

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}+\sqrt{c}}+\frac{\sqrt{b}}{\sqrt{b}+\sqrt{a}+\sqrt{c}}+\frac{\sqrt{c}}{\sqrt{c}+\sqrt{a}+\sqrt{b}}=1.$$

Egalitatea are loc dacă $a=\sqrt{bc}$, $b=\sqrt{ca}$ şi $c=\sqrt{ab}$, adică $a=b=c$.

Soluția 2: Folosind inegalitatea mediilor obținem

$$\frac{a}{a+\sqrt{(a+b)(a+c)}}=\frac{a(\sqrt{(a+b)(a+c)}-a)}{(a+b)(a+c)-a^2}\leq\frac{a\cdot\frac{a+b+a+c}{2}-a^2}{ab+bc+ca}=\frac{ab+ac}{2(ab+bc+ca)}.$$

Scriind încă două relații similare, prin adunare, rezultă inegalitatea dorită.

Soluția 3: (*Titu Zvonaru*)
Cu inegalitatea Cauchy-Buniakovski-Schwarz obținem $(a+b)(a+c)\geq(a+\sqrt{bc})^2$ și atunci este suficient să demonstrăm că

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}}\leq 1$$

sau

$$\frac{\sqrt{bc}}{2a+\sqrt{bc}}+\frac{\sqrt{ca}}{2b+\sqrt{ca}}+\frac{\sqrt{ab}}{2c+\sqrt{ab}}\geq 1.$$

Folosind inegalitatea lui Bergström, rezultă

$$\frac{\sqrt{bc}}{2a+\sqrt{bc}}+\frac{\sqrt{ca}}{2b+\sqrt{ca}}+\frac{\sqrt{ab}}{2c+\sqrt{ab}}=\frac{bc}{2a\sqrt{bc}+bc}+\frac{ca}{2b\sqrt{ca}+ca}+\frac{ab}{2c\sqrt{ab}+ab}\geq$$

$$\geq\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}+ab+bc+ca}=1.$$

Soluţia 4: (*Marius Valentin Drăgoi*)
Ca mai sus, este suficient să demonstrăm că

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}}\le 1.$$

Din inegalitatea dintre media geometrică şi cea armonică avem

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}}\le \frac{a}{2a+\frac{2bc}{b+c}}+\frac{b}{2b+\frac{2ca}{c+a}}+\frac{c}{2c+\frac{2ab}{a+b}}=$$

$$\frac{a(b+c)}{2(ab+bc+ca)}+\frac{b(c+a)}{2(ab+bc+ca)}+\frac{c(a+b)}{2(ab+bc+ca)}=1.$$

Soluţia 5: Inegalitatea fiind omogenă, este suficient să o demonstrăm în cazul $ab+bc+ca=3$ (modificând proporţional variabilele, putem reduce mereu problema la acest caz).
În acest caz, inegalitatea de demonstrat revine la

$$\frac{a}{a+\sqrt{a^2+3}}+\frac{b}{b+\sqrt{b^2+3}}+\frac{c}{c+\sqrt{c^2+3}}\le 1.$$

Dar din inegalitatea dintre media pătratică şi cea aritmetică avem că

$$\sqrt{\frac{a^2+1+1+1}{4}}\ge\frac{a+1+1+1}{4}$$

şi analoagele, adică $\sqrt{a^2+3}\ge\dfrac{a+3}{2}$. Este, deci, suficient să demonstrăm că

$$\frac{a}{a+\dfrac{a+3}{2}}+\frac{b}{b+\dfrac{b+3}{2}}+\frac{c}{c+\dfrac{c+3}{2}}\le 1,$$

adică

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\le\frac{3}{2}$$

(în condiţiile egalităţii $ab+bc+ca=3$). Eliminând numitorii, se ajunge la

$$6abc+4(ab+bc+ca)+2(a+b+c)\le 3abc+3(ab+bc+ca)+3(a+b+c)+3.$$

Folosind condiţia $ab+bc+ca=3$, este suficient să arătăm că $3abc\le a+b+c$, adică $9abc\le(ab+bc+ca)(a+b+c)$. Aceasta rezultă imediat aplicând inegalitatea dintre media aritmetică şi cea geometrică pentru cele trei numere din fiecare paranteză.

Multe mulţumiri lui *Titu Zvonaru* care mi-a semnalat sursa problemei şi trei dintre soluţiile prezentate mai sus.

Am primit soluţii de la: *Titu Zvonaru, Marius Valentin Drăgoi, Marian Cucoaneş, Ionuţ Mihai Drăgoi, Adrian Zanca* şi *Ioan Codreanu.*

Problem of the week no. 338

Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+a)(b+c)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}} \le 1.$$

Walther Janous, Crux Mathematicorum, june 1991

Solution 1: We have $\sqrt{(x+y)(x+z)} \ge \sqrt{xy}+\sqrt{xz}$, $\forall\, x, y, z > 0$.
Indeed, by squaring, the previous inequality becomes $x^2+yz \ge 2x\sqrt{yz}$, i.e.
$(x-\sqrt{yz})^2 \ge 0$, obviously true.
Using this inequality we get

$$\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+a)(b+c)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}} \le$$

$$\frac{a}{a+\sqrt{ab}+\sqrt{ac}}+\frac{b}{b+\sqrt{ba}+\sqrt{bc}}+\frac{c}{c+\sqrt{ca}+\sqrt{cb}} =$$

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}+\sqrt{c}}+\frac{\sqrt{b}}{\sqrt{b}+\sqrt{a}+\sqrt{c}}+\frac{\sqrt{c}}{\sqrt{c}+\sqrt{a}+\sqrt{b}} = 1.$$

Equality holds when $a=\sqrt{bc}$, $b=\sqrt{ca}$ and $c=\sqrt{ab}$, i.e. $a=b=c$.

Soluția 2: Using AM-GM we have

$$\frac{a}{a+\sqrt{(a+b)(a+c)}} = \frac{a(\sqrt{(a+b)(a+c)}-a)}{(a+b)(a+c)-a^2} \le \frac{a\cdot\frac{a+b+a+c}{2}-a^2}{ab+bc+ca} = \frac{ab+ac}{2(ab+bc+ca)}.$$

Adding this to two similar inequalities leads to the desired result.

Solution 3: (*Titu Zvonaru*)
From CBS we have $(a+b)(a+c) \ge (a+\sqrt{bc})^2$ so it is sufficient to prove that

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}} \le 1$$

or

$$\frac{\sqrt{bc}}{2a+\sqrt{bc}}+\frac{\sqrt{ca}}{2b+\sqrt{ca}}+\frac{\sqrt{ab}}{2c+\sqrt{ab}} \ge 1.$$

Using Titu's Lemma, we obtain

$$\frac{\sqrt{bc}}{2a+\sqrt{bc}}+\frac{\sqrt{ca}}{2b+\sqrt{ca}}+\frac{\sqrt{ab}}{2c+\sqrt{ab}}=\frac{bc}{2a\sqrt{bc}+bc}+\frac{ca}{2b\sqrt{ca}+ca}+\frac{ab}{2c\sqrt{ab}+ab}\geq$$
$$\geq\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}+ab+bc+ca}=1.$$

Solution 4: (*Marius Valentin Drăgoi*)
As above, it is sufficient to prove that

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}}\leq 1.$$

From the inequality between the geometric and harmonic mean we have

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}}\leq\frac{a}{2a+\frac{2bc}{b+c}}+\frac{b}{2b+\frac{2ca}{c+a}}+\frac{c}{2c+\frac{2ab}{a+b}}=$$

$$\frac{a(b+c)}{2(ab+bc+ca)}+\frac{b(c+a)}{2(ab+bc+ca)}+\frac{c(a+b)}{2(ab+bc+ca)}=1.$$

Solution 5: The inequality being homogeneous, it is sufficient to prove it in the case when $ab+bc+ca=3$ (by scaling the variables, we can always reduce the general case to this case).
Thus, the inequality to be proven reduces to

$$\frac{a}{a+\sqrt{a^2+3}}+\frac{b}{b+\sqrt{b^2+3}}+\frac{c}{c+\sqrt{c^2+3}}\leq 1.$$

But from the inequality between the quadratic mean and the arithmetic mean, we have

$$\sqrt{\frac{a^2+1+1+1}{4}}\geq\frac{a+1+1+1}{4}$$

and its analogues, i.e. $\sqrt{a^2+3}\geq\dfrac{a+3}{2}$ and analogues. It is therefore sufficient to show that

$$\frac{a}{a+\dfrac{a+3}{2}}+\frac{b}{b+\dfrac{b+3}{2}}+\frac{c}{c+\dfrac{c+3}{2}}\leq 1,$$

i.e.

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\leq\frac{3}{2}$$

(under the condition $ab+bc+ca=3$). Clearing denominators, we get to

$$6abc+4(ab+bc+ca)+2(a+b+c)\leq 3abc+3(ab+bc+ca)+3(a+b+c)+3.$$

Using the condition $ab+bc+ca=3$, it is sufficient to prove that $3abc\leq a+b+c$, i.e. $9abc\leq(ab+bc+ca)(a+b+c)$. This follows immediately from the AM-GM inequality for the three numbers within each of the two parenthesis.