SOLUTIONS

Problem 1, junior level. Find all integers $n \geq 0$ for which there exist integers a and b such that $a + 2^b = n^{2022}$ and $a^2 + 4^b = n^{2023}$.

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Solution. The only number is $n = 1$.

To this end, notice that $2(a^2 + 4^b) \geq (a + 2^b)^2$ to conclude that $2n^{2023} \geq n^{4044}$, hence $n = 0$ or $n = 1$. The case $n = 0$ leads to no solution, while $n = 1$ holds for $a = 0$ and $b = 0$.

Problem 2, junior level. Let $n \in \mathbb{N}$, $n \geq 4$ and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_k^3 = a_{k+1}^2 + a_{k+2}^2 + a_{k+3}^2$$

for all $k \in \{1, 2, \ldots, n\}$ – indices are considered modulo n. Show that $a_1 = a_2 = \ldots = a_n$.

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Solution. Suppose there exists $i \in \{1, 2, \ldots, n\}$ for which $a_i = 0$. Clearly $a_{i+1} = 0$, and consequently $a_1 = a_2 = \ldots = a_n = 0$.

Suppose now that $a_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$. Choose $p, q \in \{1, 2, \ldots, n\}$ such that $a_q \leq a_i \leq a_p$ for all $i \in \{1, 2, \ldots, n\}$.

Now $a_p^3 = a_{p+1}^2 + a_{p+2}^2 + a_{p+3}^2 \leq 3a_p^2$ and $a_q^3 = a_{q+1}^2 + a_{q+2}^2 + a_{q+3}^2 \geq 3a_q^2$ yield $a_p \leq 3 \leq a_q$, therefore $a_1 = a_2 = \ldots = a_n = 3$.

The proof is complete.

Problem 3, junior level & problem 1, senior level. A *domino* is a rectangle formed by two unit squares that share a common side. A number of 18 dominoes fit together to tile a 6×6 square. Show that some line crossing the interior of the square crosses the interior of no domino. Is it possible that such a line be unique?

Solution. Let the square be $[0, 6] \times [0, 6]$. We first show that either some grid-vertical $x = i$, $i = 1, 2, 3, 4, 5$, or some grid-horizontal $y = j$, $j = 1, 2, 3, 4, 5$, crosses no tile. Let m_i and n_j be the number of tiles crossed by the grid-vertical $x = i$ and the grid-horizontal $y = j$, respectively. Clearly, a grid-vertical crosses only horizontal tiles, and a grid-horizontal crosses only vertical tiles.

Since every horizontal tile is crossed by a single grid-vertical, the number of horizontal tiles is $m_1 + m_2 + m_3 + m_4 + m_5$. Similarly, the number of vertical tiles is $n_1 + n_2 + n_3 + n_4 + n_5$. Hence the number of tiles is $m_1 + m_2 + m_3 + m_4 + m_5 + n_1 + n_2 + n_3 + n_4 + n_5 = 18$. Consequently, either some $m_i \leq 1$, $i = 1, 2, 3, 4, 5$, or some $n_j \leq 1$, $j = 1, 2, 3, 4, 5$.

Now, for each positive integer $i \leq 5$, the rectangle $[0, i] \times [0, 6]$ consists of a certain number of tiles and m_i unit cells, the left halves of the horizontal tiles the grid-vertical $x = i$ crosses. Since the area of each $[0, i] \times [0, 6]$ and the area of each tile are both even, so is each m_i. Similarly, each n_j is even.

Finally, by the conclusion of the preceding paragraph, either some $m_i = 0$, $i = 1, 2, 3, 4, 5$, in which case the corresponding grid-vertical $x = i$ crosses no tile; or some $n_j = 0$, $j = 1, 2, 3, 4, 5$, in which case the corresponding grid-horizontal $y = j$ crosses no tile.

Alternative Approach. Suppose, in the above setting, that every grid-line, whether vertical or horizontal, crosses at least one tile. Then the five m_i and the five n_j are all positive even integers, i. e., they are all at least 2. Consequently, the ten add up to at least $20 > 18$ which is a contradiction. This establishes the first part.

The answer to the second part is in the affirmative. To prove this, we exhibit a domino tiling of the square $[0, 6] \times [0, 6]$ with a single separating line, i. e., one crossing no tile. Clearly, grid-lines alone are to be considered.

Begin by tiling the rectangle $[0, 3] \times [0, 6]$ by four horizontal dominoes, namely,

$$[0, 2] \times [0, 1], \quad [0, 2] \times [3, 4], \quad [1, 3] \times [4, 5], \quad [1, 3] \times [5, 6],$$

and five vertical dominoes, namely,

$$[2, 3] \times [0, 2], \quad [0, 1] \times [1, 3], \quad [1, 2] \times [1, 3], \quad [2, 3] \times [2, 4], \quad [0, 1] \times [4, 6].$$

Notice that the grid–horizontal $y = 4$ is the single separating line of this tiling.

Next, tile the rectangle $[3, 6] \times [0, 6]$ by a copy of the reflection of the above tiling in the grid-horizontal $y = 3$, to make the grid-horizontal $y = 2$ its single separating line. Explicitly, the four horizontal tiles are

$$[4, 6] \times [0, 1], \quad [4, 6] \times [1, 2], \quad [3, 5] \times [2, 3], \quad [3, 5] \times [5, 6],$$

and the five vertical tiles are

$$[3, 4] \times [0, 2], \quad [5, 6] \times [2, 4], \quad [3, 4] \times [3, 5], \quad [4, 5] \times [3, 5], \quad [5, 6] \times [4, 6].$$

The grid-horizontal $y = 2$ is clearly the single separating line of this tiling.

Finally, the two tilings fit together along the grid-vertical $x = 3$ to form an overall tiling of the square $[0, 6] \times [0, 6]$ with a single separating line — the grid-vertical $x = 3$, of course.

Remark. Consider tiling an $m \times n$ rectangle by dominoes. Clearly, for such a tiling to exist, it is necessary that mn be even — assume this from now on.

It is easily seen that for any domino tiling of a rectangle $2 \times n$, $3 \times n$ or $4 \times n$, there always exists a separating grid-line, i. e., one crossing no tile.

On the other hand, if $m \geq 5$, $n \geq 5$, and $(m, n) \neq (6, 6)$, then an $m \times n$ rectangle can be tiled by dominoes so that no grid-line is separating. This can be done by first dealing with a 5×6 and a 6×8 rectangle, respectively, to extend a tiling of an $m \times n$ rectangle with no separating grid-lines to one of an $(m + 2, n)$ rectangle.

Problem 4, junior level & problem 2, senior level. Let $ABCD$ be a convex quadrilateral and let P be a point inside such that $\angle APB + \angle CPD = \angle APD + \angle BPC$, $\angle PAD + \angle PCD =$

$\angle PAB + \angle PCB$ and $\angle PDC + \angle PBC = \angle PDA + \angle PBA$. Prove that the quadrilateral $ABCD$ is circumscriptible.

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Solution. As usual, let A, B, C, D be the measures of the angles of the quadrilateral.

Notice that $\angle PDC + \angle PBC = \angle PDA + \angle PBA$ implies $\angle PDA + \angle PBA = (B + D)/2$. In the same manner, $\angle PAB + \angle PCB = (A + C)/2$. Add the last equalities to obtain $180° = (A+B+C+D)/2 = (\angle PBA + \angle PAB) + \angle PDA + \angle PCB = (180° - \angle APB) + \angle PDA + \angle PCB$, and notice that $\angle PDA + \angle PCB = \angle APB$ shows that circles APD and BPC are tangent at P.

Let (O_1, R_1), (O_2, R_2), (O_3, R_3), (O_4, R_4) be the circles PAB, PBC, PCD and PDA respectively. Recall that points P, O_1, O_3 are collinear, and similarly, points P, O_2, O_4 are collinear. Further, $\angle O_2O_1O_4 + \angle O_2O_3O_4 = 180° - \angle APB + 180° - \angle CPD = 180°$, so $O_1O_2O_3O_4$ is a cyclic quadrilateral; let R be its circumradius.

By Sine Law,

$$2R = \frac{O_1O_3}{\sin O_2} = \frac{O_2O_4}{\sin O_1} \Longrightarrow \frac{R_1 + R_3}{\sin \angle BPC} = \frac{R_2 + R_4}{\sin \angle APB},$$

then

$$\frac{AB}{\sin \angle APB} + \frac{CD}{\sin \angle CPD} = 2(R_1 + R_3) \Longrightarrow AB + CD = 2(R_1 + R_3) \sin \angle APB.$$

Similarly, $BC + DA = 2(R_2 + R_4) \sin \angle BPC$. All the above lead to $AB + CD = BC + DA$, hence the claim.