Problema săptămânii 326

Dacă $a, b, c \in (0, \infty)$, demonstraţi că

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{(a+b)^2}{a^2+bc}+\frac{(b+c)^2}{b^2+ca}+\frac{(c+a)^2}{c^2+ab}.$$

Florin Rotaru, GM nr. 1/2022

Soluţia 1:

Din inegalitatea mediilor rezultă $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$, deci este suficient să demonstrăm că

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \geq \frac{(a+b)^2}{a^2+bc}+\frac{(b+c)^2}{b^2+ca}+\frac{(c+a)^2}{c^2+ab}.$$

Din inegalitatea Cauchy-Buniakowsky-Schwarz, forma Titu Andreescu, avem

$$1+\frac{b}{c}=\frac{a^2}{a^2}+\frac{b^2}{bc} \geq \frac{(a+b)^2}{a^2+bc},$$

care, adunată cu analoagele, duce la inegalitatea cerută.
Egalitatea are loc dacă $a=b=c$.

Soluţia 2: Notăm $\frac{a}{b}=x^3$, $\frac{b}{c}=y^3$, $\frac{c}{a}=z^3$. Atunci $x, y, z > 0$ şi $xyz=1$.
Inegalitatea devine $2(x^3+y^3+z^3) \geq \frac{(x^3+1)^2}{x^6+x^3z^3}+\frac{(y^3+1)^2}{y^6+y^3x^3}+\frac{(z^3+1)^2}{z^3+z^3y^3}$. Dar

$$\frac{(x^3+1)^2}{x^6+x^3z^3}=\frac{(x^3+xyz)^2}{x^6+x^3z^3}\cdot xyz=\frac{(x^2+yz)^2yz}{x^3+z^3}=\frac{x^4yz+2x^2y^2z^2+y^3z^3}{x^3+z^3} \leq$$

$$\frac{x^4yz+xyz^4+x^3y^3+y^3z^3}{x^3+z^3}=xyz+y^3,$$

deci inegalitatea de demonstrat rezultă din $2(x^3+y^3+z^3) \geq x^3+y^3+z^3+3xyz$ care rezultă imediat din inegalitatea mediilor.
Egalitatea are loc dacă $x=y=z=1$, adică $a=b=c$.

Am primit soluţii de la: *Marius Valentin Drăgoi, Marian Cucoaneş* şi *Alberto Radu.*

Problem of the week no. 326

If $a, b, c \in (0, \infty)$, prove that

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{(a+b)^2}{a^2+bc}+\frac{(b+c)^2}{b^2+ca}+\frac{(c+a)^2}{c^2+ab}.$$

Florin Rotaru, GM nr. 1/2022

Solution 1:

From the AM-GM inequality we get $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$, therefore it is sufficient to prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \geq \frac{(a+b)^2}{a^2+bc}+\frac{(b+c)^2}{b^2+ca}+\frac{(c+a)^2}{c^2+ab}.$$

From CBS (Titu's Lemma),

$$1+\frac{b}{c}=\frac{a^2}{a^2}+\frac{b^2}{bc} \geq \frac{(a+b)^2}{a^2+bc},$$

which, added to its analogues, leads to the desired conclusion.
Equality holds when $a=b=c$.

Solution 2: Put $\frac{a}{b}=x^3$, $\frac{b}{c}=y^3$, $\frac{c}{a}=z^3$. Then $x, y, z > 0$ and $xyz=1$.
The inequality becomes $2(x^3+y^3+z^3) \geq \frac{(x^3+1)^2}{x^6+x^3z^3}+\frac{(y^3+1)^2}{y^6+y^3x^3}+\frac{(z^3+1)^2}{z^3+z^3y^3}$. But

$$\frac{(x^3+1)^2}{x^6+x^3z^3}=\frac{(x^3+xyz)^2}{x^6+x^3z^3}\cdot xyz=\frac{(x^2+yz)^2yz}{x^3+z^3}=\frac{x^4yz+2x^2y^2z^2+y^3z^3}{x^3+z^3} \leq$$

$$\frac{x^4yz+xyz^4+x^3y^3+y^3z^3}{x^3+z^3}=xyz+y^3,$$

and our inequality follows from $2(x^3+y^3+z^3) \geq x^3+y^3+z^3+3xyz$ which is true by AM-GM.
Equality holds when $x=y=z=1$, i.e. $a=b=c$.