Problema săptămânii 317:

Fie $ABCD$ un patrulater inscriptibil si K, L, M şi N – mijloacele laturilor $[AB]$, $[BC]$, $[CD]$ şi respectiv $[AD]$. Arătaţi că ortocentrele triunghiurilor AKN, BKL, CLM şi DMN sunt vârfurile unui paralelogram.

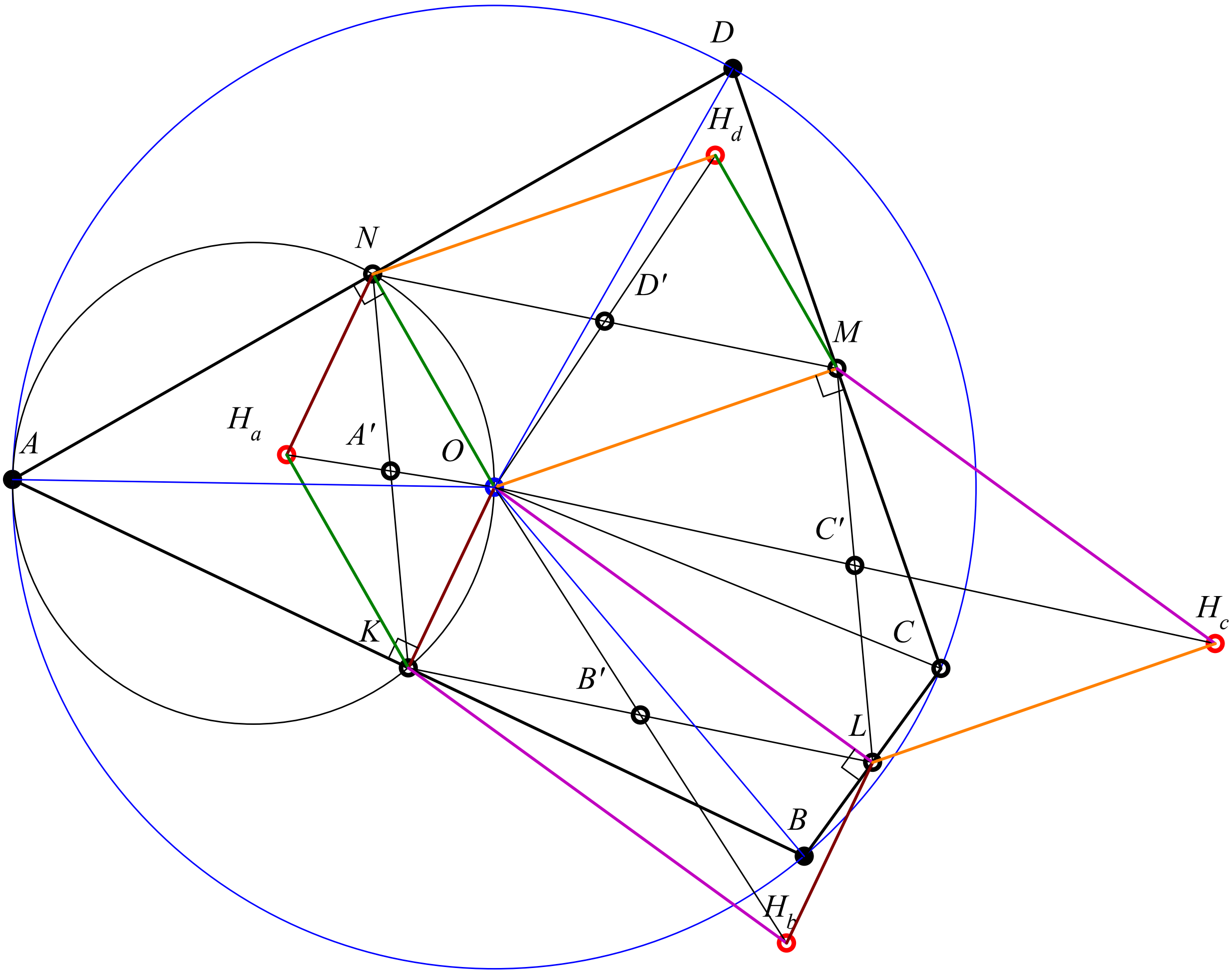

Fig.1

SOLUŢIE (*Mihai Miculiţa*): Notând acum cu: O – centrul cercului circumscris patrulaterului inscriptibil $ABCD$, cu H_a, H_b, H_c şi H_d – ortocentrele triunghiurilor AKN, BKL, CLM şi respectiv DMN; iar cu: $\{A'\} := [OH_a] \cap [NK], \{B'\} := [OH_b] \cap [KL], \{C'\} := [OH_c] \cap [LM]$, $\{D'\} := [OH_d] \cap [MN]$ (**v.Fig.1**), din:

$$\left.\begin{array}{r}\left.\begin{array}{r}\left.\begin{array}{l}[OA] \equiv [OD] \\ [NA] \equiv [ND]\end{array}\right\} \Rightarrow ON \perp AD \\ KH_a \perp AD\end{array}\right\} \Rightarrow ON \parallel KH_a \\ \left.\begin{array}{r}\left.\begin{array}{l}[OA] \equiv [OB] \\ [KA] \equiv [KB]\end{array}\right\} \Rightarrow OK \perp AB \\ NH_a \perp AB\end{array}\right\} \Rightarrow OK \parallel NH_a\end{array}\right\} \Rightarrow ONH_aK - paralelogram.$$

În mod analog arătăm că şi patrulaterele OKH_bL, OLH_cM şi OMH_dN – sunt paralelograme; iar din:

$$\left.\begin{array}{r}ONH_aK - paralelogram \\ \{A'\} := [OH_a] \cap [NK]\end{array}\right\} \Rightarrow \left\{\begin{array}{l}[A'N] \equiv [A'K] \quad (1) \\ [A'O] \equiv [A'H_a] \quad (2)\end{array}\right.,$$

$$\left.\begin{array}{r}OKH_bL - paralelogram \\ \{B'\} := [OH_b] \cap [KL]\end{array}\right\} \Rightarrow \left\{\begin{array}{l}[B'K] \equiv [B'L] \quad (3) \\ [B'O] \equiv [B'H_b] \quad (4)\end{array}\right.,$$

$$\left.\begin{array}{r} OLH_cM - paralelogram \\ \{C'\}:=[OH_c]\cap[LM] \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} [C'L]\equiv[C'M] & (5) \\ [C'O]\equiv[C'H_c] & (6) \end{array}\right. \quad \text{şi}$$

$$\left.\begin{array}{r} OMH_dN - paralelogram \\ \{D'\}:=[OH_d]\cap[MN] \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} [D'M]\equiv[D'N] & (7) \\ [D'O]\equiv[D'H_d]. & (8) \end{array}\right.$$

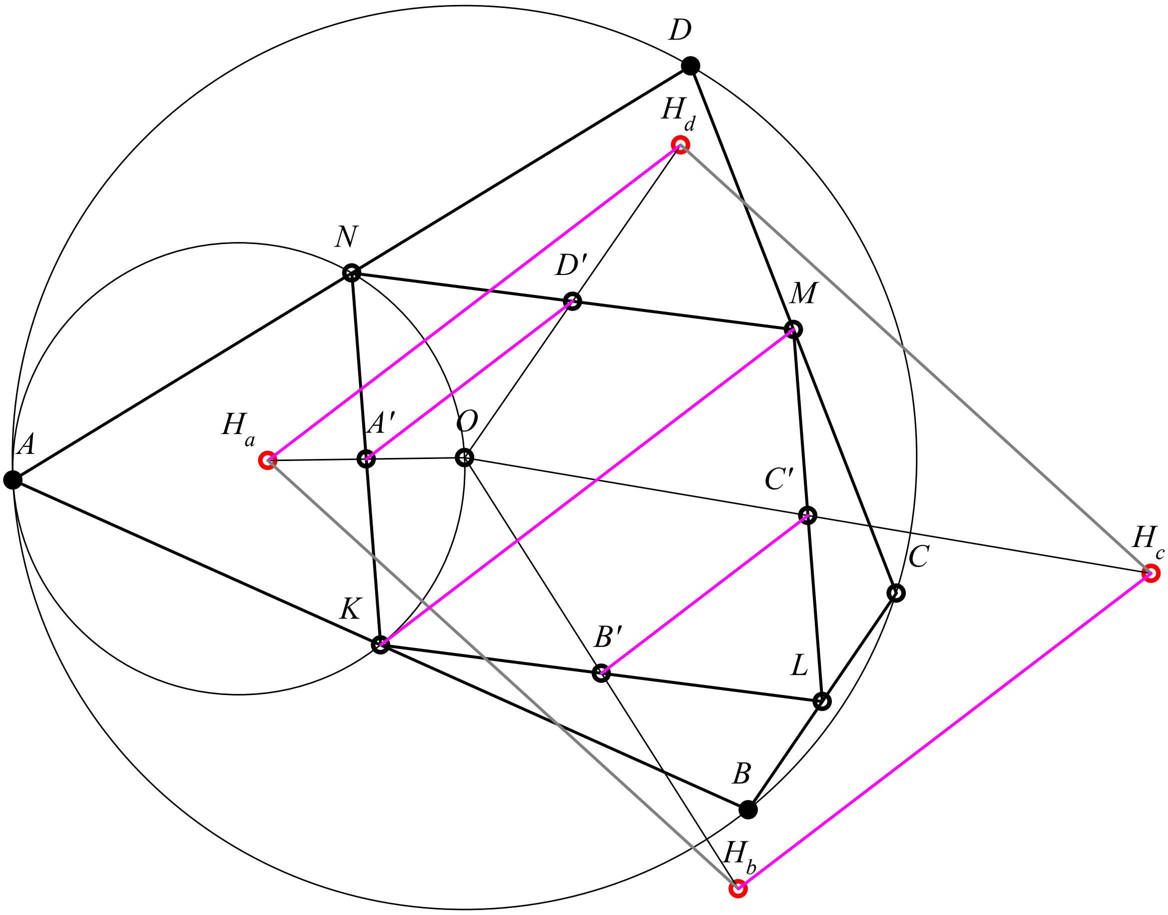

Fig.2.

În fine, din (**v.Fig.2**):

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{r} [A'O]\equiv[A'H_a](2) \\ [D'O]\equiv[D'H_d](8) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} H_aH_d \parallel A'D' \\ |H_aH_d| = 2\cdot|A'D'| \end{array}\right. \\
\left.\begin{array}{r} [A'N]\equiv[A'K](1) \\ [D'N]\equiv[D'M](7) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} A'D' \parallel KM \\ |KM| = 2\cdot|A'D'| \end{array}\right.
\end{array}\right\} \Rightarrow \left\{\begin{array}{c} H_aH_d \parallel KM \\ [H_aH_d]\equiv[KM] \end{array}\right. \\
\left.\begin{array}{l}
\left.\begin{array}{r} [B'O]\equiv[B'H_b](4) \\ [C'O]\equiv[C'H_c](6) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} H_bH_c \parallel B'C' \\ |H_bH_c| = 2\cdot|B'C'| \end{array}\right. \\
\left.\begin{array}{r} [B'K]\equiv[B'L](3) \\ [C'L]\equiv[C'M](5) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} KM \parallel B'C' \\ |KM| = 2\cdot|B'C'| \end{array}\right.
\end{array}\right\} \Rightarrow \left\{\begin{array}{c} H_bH_c \parallel KM \\ [H_bH_c]\equiv[KM] \end{array}\right.
\end{array}\right\} \Rightarrow \left\{\begin{array}{c} H_aH_d \parallel H_bH_c \\ [H_aH_d]\equiv[H_bH_c] \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{H_aH_bH_cH_d - paralelogram}. \blacksquare$$