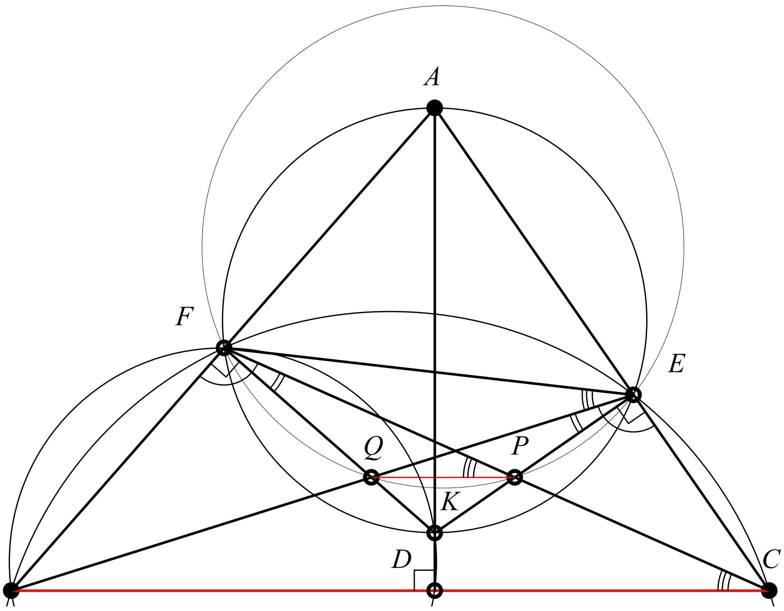
Problema săptămânii 313

Fie D piciorul înălţimii duse din vârful A al unui triunghi ascuţitunghic ABC, iar $K \in (AD)$ un punct arbitrar, diferit de ortocentrul triunghiului ABC. Notăm cu E şi F proiecţiile punctului K pe dreptele AC, respectiv AB, iar cu $\{P\} = KE \cap CF$ şi cu $\{Q\} = KF \cap BE$. Arătaţi că $PQ \parallel BC$.

Thanos Kalogerakis

Soluţie: Fie H ortocentrul triunghiului. Distingem două configuraţii, după cum $K \in (HD)$ sau $K \in (AH)$. Vom trata numai primul caz, celălalt fiind analog.



SOLUŢIE (*Mihai Miculiţa*): Avem:

$$\left.\begin{array}{l}\left.\begin{array}{l}AD \perp BC\\ KF \perp AB\end{array}\right\} \Rightarrow BDKF - inscriptibil \Rightarrow |AB|\cdot|AF| = \underline{|AD|\cdot|AK|}\\ \left.\begin{array}{l}AD \perp BC\\ KE \perp AC\end{array}\right\} \Rightarrow CEKD - inscriptibil \Rightarrow \underline{|AD|\cdot|AK|} = |AC|\cdot|AE|\end{array}\right\} \Rightarrow |AB|\cdot|AF| = |AC|\cdot|AE| \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow BCEF - inscriptibil; (1) \Rightarrow \widehat{CFB} \equiv \widehat{CEB}\\ \left.\begin{array}{l}KF \perp AB\\ KE \perp AC\end{array}\right\} \Rightarrow \widehat{KFB} \equiv \widehat{CEK}\left(=90^0\right)\end{array}\right\} \Rightarrow m\left(\widehat{PFQ}\right) = m\left(\widehat{CFB}\right) - m\left(\widehat{KFB}\right) =$$

$$\left.\begin{array}{r} = m\left(\widehat{CEB}\right) - m\left(\widehat{CEK}\right) = m\left(\widehat{PEQ}\right) \Rightarrow \widehat{PFQ} \equiv \widehat{PEQ} \Rightarrow PEFQ - inscriptibil \Rightarrow \widehat{QPF} \equiv \underline{\widehat{QEF}}\\ BCEF - inscriptibil\ (1) \Rightarrow \underline{\widehat{BEF}} \equiv \widehat{BCF}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \widehat{QPF} \equiv \widehat{BCF} \Rightarrow \boxed{PQ \parallel BC}. \blacksquare$$

Am primit şi o altă soluţie de la *Mihai Miculiţa*.

Problem of the week no. 313

Let D be the foot of the altitude dropped from A in an acute triangle ABC, and consider $K \in (AD)$ an arbitrary point, different from the orthocenter of ABC. Let E and F be the projections of point K onto the lines AC and AB, respectively, and consider $\{P\} = KE \cap CF$ and $\{Q\} = KF \cap BE$. Prove that $PQ \parallel BC$.

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Solution: (*Mihai Miculiţa*)
There are two cases, depending on the position of K with respect to the orthocenter of ABC. WE assume that K is between D and the orthocenter, the other case being similar.
Quadrilaterals $BDKF$ and $CEKD$ are cyclic. A belongs to the radical axis of their circumcircles, therefore A has equal power with respect to the two circles. It follows that $AF \cdot AB = AE \cdot AC$. By the converse of the Power of a Point Theorem, we obtain that $BCEF$ is also cyclic. Thus, $\sphericalangle BEC = \sphericalangle BFC$ and $\sphericalangle QEP = \sphericalangle BEC - 90^\circ = \sphericalangle BFC - 90^\circ = \sphericalangle QFP$, which means that $EFQP$ is also cyclic. Finally, $\sphericalangle FPQ = \sphericalangle FEQ = \sphericalangle FEB = \sphericalangle FCB$, showing that $PQ \parallel BC$.