Problema săptămânii 302

Fie a, b, c, d numere reale astfel încât $0 \le a, b, c, d \le 1$. Arătaţi că

$$ab(a-b) + bc(b-c) + cd(c-d) + da(d-a) \le \frac{8}{27}.$$

Determinaţi toate cvadrupletele (a, b, c, d) de numere reale cu proprietatea $0 \le a, b, c, d \le 1$ pentru care inegalitatea de mai sus este satisfăcută cu egalitate.

Olimpiada Benelux, 2019

Soluţie: Inegalitatea este circular simetrică deci putem, deocamdată, presupune că $a = \max\{a, b, c, d\}$.

Avem $ab(a-b) + bc(b-c) + cd(c-d) + da(d-a) = a^2(b-d) - a(b^2-d^2) - c^2(b-d) + c(b^2-d^2) = (b-d)[a^2-c^2-a(b+d)+c(b+d)] = (b-d)(a-c)(a+c-b-d)$.

Dacă $(b-d)(a-c)(a+c-b-d) \le 0$ inegalitatea din enunţ este satisfăcută, fără egalitate.

Dacă $(b-d)(a-c)(a+c-b-d) > 0$, cum $a-c \ge 0$, avem două cazuri:

1. $b-d > 0$ şi $a+c-b-d > 0$

Din inegalitatea mediilor,

$$\sqrt[3]{(b-d)(a-c)(a+c-b-d)} \le \frac{(b-d)+(a-c)+(a+c-b-d)}{3} = \frac{2a-2d}{3} \le \frac{2}{3},$$

de unde inegalitatea din enunţ.

Egalitatea are loc dacă $a-d=1$ şi $b-d = a-c = a+c-b-d = \frac{2}{3}$, adică $a=1$, $d=0$, $c=\frac{1}{3}$, $b=\frac{2}{3}$.

2. $b-d < 0$ şi $a+c-b-d < 0$

Din inegalitatea mediilor,

$$\sqrt[3]{(d-b)(a-c)(b+d-a-c)} \le \frac{(d-b)+(a-c)+(b+d-a-c)}{3} = \frac{2d-2c}{3} \le \frac{2}{3},$$

de unde inegalitatea din enunţ.

Egalitatea are loc dacă $d-c=1$ şi $d-b = a-c = b+d-a-c = \frac{2}{3}$, adică $d=1$, $c=0$, $a=\frac{2}{3}$, $b=\frac{1}{3}$, care însă nu verifică $a = \max\{a, b, c, d\}$, deci în acest caz nu avem egalitate.

Aşadar inegalitatea a fost demonstrată.

Renunţând acum la condiţia $a = \max\{a, b, c, d\}$ găsim că egalitatea are loc pentru $a=1$, $d=0$, $c=\frac{1}{3}$, $b=\frac{2}{3}$ şi pentru cele trei permutări circulare ale acestui cvadruplet.

Am primit soluţii de la: *David Ghibu, Denis Nica, Emanuel Mazăre* şi *Aida Mitroi.*

Problem of the week no. 302

Let a, b, c, d be real numbers with $0 \leq a, b, c, d \leq 1$. Prove that

$$ab(a-b)+bc(b-c)+cd(c-d)+da(d-a) \leq \frac{8}{27}.$$

Find all quadruples (a, b, c, d) of real numbers with $0 \leq a, b, c, d \leq 1$ for which equality holds in the above inequality.

Benelux Mathematical Olympiad, 2019

Solution: Because of the circular symmetry of the inequality, we may assume for now that $a = \max\{a, b, c, d\}$.

We have $ab(a-b)+bc(b-c)+cd(c-d)+da(d-a) = a^2(b-d)-a(b^2-d^2)-c^2(b-d)+c(b^2-d^2) = (b-d)[a^2-c^2-a(b+d)+c(b+d)] = (b-d)(a-c)(a+c-b-d)$.

If $(b-d)(a-c)(a+c-b-d) \leq 0$ the inequality is clearly satisfied, with no equality in this case.

If $(b-d)(a-c)(a+c-b-d) > 0$, as $a-c \geq 0$, we ave two cases:

1. $b-d > 0$ and $a+c-b-d > 0$

From the AM-GM inequality,

$$\sqrt[3]{(b-d)(a-c)(a+c-b-d)} \leq \frac{(b-d)+(a-c)+(a+c-b-d)}{3} = \frac{2a-2d}{3} \leq \frac{2}{3},$$

and our inequality follows immediately. Equality holds if $a-d=1$ and $b-d = a-c = a+c-b-d = \frac{2}{3}$, i.e. $a=1$, $d=0$, $c=\frac{1}{3}$, $b=\frac{2}{3}$.

2. $b-d < 0$ and $a+c-b-d < 0$

From the AM-GM inequality,

$$\sqrt[3]{(d-b)(a-c)(b+d-a-c)} \leq \frac{(d-b)+(a-c)+(b+d-a-c)}{3} = \frac{2d-2c}{3} \leq \frac{2}{3},$$

and our inequality follows.

Equality would hold if $d-c=1$ and $d-b=a-c=b+d-a-c=\frac{2}{3}$, i.e. $d=1$, $c=0$, $a=\frac{2}{3}$, $b=\frac{1}{3}$, witch does not satisfy the condition $a=\max\{a, b, c, d\}$, therefore equality is impossible in this case.

Removing the condition $a = \max\{a, b, c, d\}$ we find that equality holds for $a=1$, $d=0$, $c=\frac{1}{3}$, $b=\frac{2}{3}$ and for the three circular permutations of this quadruple.

You can also find a slightly different solution on AoPS.