Problema săptămânii 290

Dacă a, b, c sunt numere reale pozitive, demonstraţi că

$$\frac{(1+a)^2(1+b)^2}{1+c^2}+\frac{(1+b)^2(1+c)^2}{1+a^2}+\frac{(1+c)^2(1+a)^2}{1+b^2} \geq 8(a+b+c).$$

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Soluţie: Cu inegalitatea mediilor obţinem

$$(1+a)^2(1+b)^2=[(1+ab)+(a+b)]^2 \geq 4(1+ab)(a+b)=4a(1+b^2)+4b(1+a^2).$$

Scriind încă două relaţii similare şi adunând se obţine

$$\frac{(1+a)^2(1+b)^2}{1+c^2}+\frac{(1+b)^2(1+c)^2}{1+a^2}+\frac{(1+c)^2(1+a)^2}{1+b^2} \geq$$

$$4a\left(\frac{1+b^2}{1+c^2}+\frac{1+c^2}{1+b^2}\right)+4b\left(\frac{1+c^2}{1+a^2}+\frac{1+a^2}{1+c^2}\right)+4c\left(\frac{1+a^2}{1+b^2}+\frac{1+b^2}{1+a^2}\right) \geq 8(a+b+c).$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

Am primit soluţii de la: *Radu Stoleriu, Emanuel Mazăre, David Ghibu* şi *Denis Nica.*

Problem of the week no. 290

If a, b, c are positive real numbers, prove that

$$\frac{(1+a)^2(1+b)^2}{1+c^2}+\frac{(1+b)^2(1+c)^2}{1+a^2}+\frac{(1+c)^2(1+a)^2}{1+b^2} \geq 8(a+b+c).$$

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Solution: Using the AM-GM inequality we get

$$(1+a)^2(1+b)^2=[(1+ab)+(a+b)]^2 \geq 4(1+ab)(a+b)=4a(1+b^2)+4b(1+a^2).$$

Writing down two more relations, similar to the one above, and adding them leads to

$$\frac{(1+a)^2(1+b)^2}{1+c^2}+\frac{(1+b)^2(1+c)^2}{1+a^2}+\frac{(1+c)^2(1+a)^2}{1+b^2} \geq$$

$$4a\left(\frac{1+b^2}{1+c^2}+\frac{1+c^2}{1+b^2}\right)+4b\left(\frac{1+c^2}{1+a^2}+\frac{1+a^2}{1+c^2}\right)+4c\left(\frac{1+a^2}{1+b^2}+\frac{1+b^2}{1+a^2}\right) \geq 8(a+b+c).$$

Equality holds if and only if $a=b=c$.