Problema săptămânii 2778

Se consideră numerele reale pozitive $a_1, a_2, \ldots, a_n$, unde $n \geq 3$, astfel încât

$$a_1 + a_2 + \ldots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}.$$

Demonstraţi că

$$\sum_{i<j} a_i a_j \geq \frac{n(n-1)}{2}.$$

Andrei Eckstein, Leonard Giugiuc, Crux Mathematicorum, Vol.47 (6), iunie 2021

Soluţia 1: Inegalitatea de demonstrat este echivalentă cu

$$\sum_{i<j} a_i a_j \cdot \sum_{k=1}^{n} \frac{1}{a_k} \geq \frac{n(n-1)}{2} \cdot \sum_{k=1}^{n} a_k.$$

Aceasta din urmă revine, după înmulţire cu $a_1 \cdot a_2 \cdot \ldots \cdot a_n$, la $[2, 2, 1, \ldots, 1, 0] \geq [2, 1, 1, \ldots, 1, 1]$, care rezultă din inegalitatea lui Muirhead.

Soluţia 2: (*Marian Cucoaneş, David Ghibu*)
Vom folosi următoarea inegalitate: dacă $x, y, z > 0$, atunci

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq x + y + z. \qquad (*)$$

Demonstraţie: Eliminând numitorii se ajunge la $(xy)^2 + (yz)^2 + (zx)^2 \geq x^2yz + xy^2z + xyz^2$ care este un caz particular al cunoscutei inegalităţi $a^2 + b^2 + c^2 \geq ab + bc + ca$.
Revenind la inegalitatea din enunţ, avem

$$\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \left(\sum_{1 \leq i < j \leq n} a_i a_j\right) = (n-1) \sum_{j=1}^{n} a_j + \sum_{1 \leq i < j < k \leq n} \left(\frac{a_j a_k}{a_i} + \frac{a_k a_i}{a_j} + \frac{a_i a_j}{a_k}\right) \overset{(*)}{\geq}$$

$$(n-1) \sum_{j=1}^{n} a_j + \sum_{1 \leq i < j < k \leq n} (a_i + a_j + a_k) = (n-1) \sum_{j=1}^{n} a_j + \frac{(n-1)(n-2)}{2} \sum_{j=1}^{n} a_j =$$

$$\frac{n(n-1)}{2} \sum_{j=1}^{n} a_j.$$

Cum $\displaystyle\sum_{j=1}^{n} \frac{1}{a_j} = \sum_{j=1}^{n} a_j$, obţinem inegalitatea dorită.
Deoarece în $(*)$ avem egalitate dacă şi numai dacă $x = y = z$, în inegalitatea dată avem egalitate dacă $a_1 = a_2 = \ldots = a_n = 1$.

Soluţia 3: (*Radu Stoleriu, David Ghibu*)
Inegalitatea de demonstrat este echivalentă cu

$$\left(\sum_{1\le i<j\le n} a_i a_j\right)\left(\sum_{j=1}^{n}\frac{1}{a_j}\right) \ge \frac{n(n-1)}{2}\cdot\sum_{j=1}^{n} a_j$$

sau

$$\frac{1}{2}\left(\sum_{i=1}^{n} a_i\cdot\sum_{j\ne i} a_j\right)\left(\sum_{j=1}^{n}\frac{1}{a_j}\right) \ge \frac{n(n-1)}{2}\cdot\sum_{j=1}^{n} a_j.$$

Dar $a_i\cdot\left(\sum_{j\ne i} a_j\right)\left(\sum_{j=1}^{n}\frac{1}{a_j}\right) = \sum_{j\ne i} a_j + a_i\cdot\sum_{j\ne i} a_j\cdot\sum_{j\ne i}\frac{1}{a_j} \ge \sum_{j\ne i} a_j + (n-1)^2 a_i.$

Sumând după $i\in\{1,2,\ldots,n\}$ relaţiile de mai sus se obţine inegalitatea cerută.

Am primit soluţii de la: *Ioan Codreanu, Marian Cucoaneş, Radu Stoleriu* şi *David Ghibu* (două soluţii).

Problem of the week no. 278

Positive real numbers $a_1, a_2, \ldots, a_n$, where $n\ge 3$, satisfy

$$a_1 + a_2 + \ldots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}.$$

Prove that

$$\sum_{i<j} a_i a_j \ge \frac{n(n-1)}{2}.$$

Andrei Eckstein, Leonard Giugiuc, Crux Mathematicorum, Vol.47 (6), june 2021

Solution 1: The inequality to be proven is equivalent to

$$\sum_{i<j} a_i a_j\cdot\sum_{k=1}^{n}\frac{1}{a_k} \ge \frac{n(n-1)}{2}\cdot\sum_{k=1}^{n} a_k.$$

The latter one becomes, after multiplication by $a_1\cdot a_2\cdot\ldots\cdot a_n$, the inequality $[2,2,1,\ldots,1,0]\ge[2,1,1,\ldots,1,1]$, which follows from Muirhead.

Solution 2: (*Marian Cucoaneş, David Ghibu*)
We use the following inequality: if $x,y,z>0$, then

$$\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z} \ge x+y+z. \qquad (*)$$

Proof: Clearing denominators leads to $(xy)^2 + (yz)^2 + (zx)^2 \geq x^2yz + xy^2z + xyz^2$ which is a particular case of $a^2 + b^2 + c^2 \geq ab + bc + ca$.
Returning to the inequality from the statement, we have

$$\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \left(\sum_{1 \leq i < j \leq n} a_i a_j\right) = (n-1)\sum_{j=1}^{n} a_j + \sum_{1 \leq i < j < k \leq n} \left(\frac{a_j a_k}{a_i} + \frac{a_k a_i}{a_j} + \frac{a_i a_j}{a_k}\right) \overset{(*)}{\geq}$$

$$(n-1)\sum_{j=1}^{n} a_j + \sum_{1 \leq i < j < k \leq n} (a_i + a_j + a_k) = (n-1)\sum_{j=1}^{n} a_j + \frac{(n-1)(n-2)}{2}\sum_{j=1}^{n} a_j =$$

$$\frac{n(n-1)}{2}\sum_{j=1}^{n} a_j.$$

As $\displaystyle\sum_{j=1}^{n} \frac{1}{a_j} = \sum_{j=1}^{n} a_j$, we obtain the desired inequality.
In $(*)$ equality holds iff $x = y = z$, therefore in our inequality we have equality iff $a_1 = a_2 = \ldots = a_n = 1$.

Solution 3: (*Radu Stoleriu, David Ghibu*)
The inequality can be written

$$\left(\sum_{1 \leq i < j \leq n} a_i a_j\right) \left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq \frac{n(n-1)}{2} \cdot \sum_{j=1}^{n} a_j$$

or

$$\frac{1}{2}\left(\sum_{i=1}^{n} a_i \cdot \sum_{j \neq i} a_j\right) \left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq \frac{n(n-1)}{2} \cdot \sum_{j=1}^{n} a_j.$$

But $\displaystyle a_i \cdot \left(\sum_{j \neq i} a_j\right) \left(\sum_{j=1}^{n} \frac{1}{a_j}\right) = \sum_{j \neq i} a_j + a_i \cdot \sum_{j \neq i} a_j \cdot \sum_{j \neq i} \frac{1}{a_j} \geq \sum_{j \neq i} a_j + (n-1)^2 a_i$.
Summing after $i \in \{1, 2, \ldots, n\}$ the inequalities above yields the inequality to be proven.