2021 STARS OF MATHEMATICS — SOLUTIONS (Junior Grade)

Problem 1. Let a_1, a_2, a_3, a_4 be positive real numbers satisfying

$$a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 = 1.$$

Show that

$$\frac{a_1a_2}{1+a_3a_4} + \frac{a_1a_3}{1+a_2a_4} + \frac{a_1a_4}{1+a_2a_3} + \frac{a_2a_3}{1+a_1a_4} + \frac{a_2a_4}{1+a_1a_3} + \frac{a_3a_4}{1+a_1a_2} \geq \frac{6}{7}.$$

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Solution. Write the generic summand in the form $a_i^2a_j^2/(a_ia_j + a_1a_2a_3a_4)$, $1 \leq i < j \leq 4$, then use the condition $\sum_{1\leq i<j\leq 4} a_ia_j = 1$ and the Cauchy-Schwarz inequality, to write

$$\begin{aligned}\sum_{1\leq i<j\leq 4} \frac{a_i^2a_j^2}{a_ia_j + a_1a_2a_3a_4} &= \frac{\sum_{1\leq i<j\leq 4}(a_ia_j + a_1a_2a_3a_4)}{1+6a_1a_2a_3a_4} \cdot \sum_{1\leq i<j\leq 4} \frac{a_i^2a_j^2}{a_ia_j + a_1a_2a_3a_4} \\ &\geq \frac{1}{1+6a_1a_2a_3a_4} \cdot \sum_{1\leq i<j\leq 4} a_ia_j = \frac{1}{1+6a_1a_2a_3a_4}\end{aligned}$$

Next, $(a_1a_2a_3a_4)^3 = a_1a_2 \cdot a_1a_3 \cdot a_1a_4 \cdot a_2a_3 \cdot a_2a_4 \cdot a_3a_4 \leq \left(\frac{1}{6}\sum_{1\leq i<j\leq 4} a_ia_j\right)^6 = \frac{1}{6^6}$, by the AM-GM inequality, so $a_1a_2a_3a_4 \leq \frac{1}{6^2}$.

Consequently,

$$\sum_{1\leq i<j\leq 4} \frac{a_i^2a_j^2}{a_ia_j + a_1a_2a_3a_4} \geq \frac{1}{1+6a_1a_2a_3a_4} \geq \frac{1}{1+6\cdot\frac{1}{6^2}} = \frac{6}{7}.$$

Clearly, equality holds if and only if $a_1 = a_2 = a_3 = a_4 = \frac{1}{\sqrt{6}}$.

Problem 2. Let ABC be a triangle, let I be its incentre and let D be the orthogonal projection of I on BC. The circle ABC crosses the line AI again at M, and the line DM again at N. Prove that the lines AN and IN are perpendicular.

Freddie Illingworth and Dominic Yeo

Solution. Alternatively, but equivalently, we show that N lies on the circle on diameter AI. This circle crosses the sides AC and AB at E and F, respectively.

Clearly, D, E and F are the contact points of the incircle of the triangle ABC with the sides BC, CA and AB, respectively, so $BD = BF$ and $CD = CE$.

On the other hand, MN is the internal angle bisectrix of $\angle BNC$, so $NB/NC = DB/DC = BF/CE$. (The last equality holds by the preceding paragraph.)

Next, read angles from the circle ABC to write $\angle FBN = \angle ABN = \angle ACN = \angle ECN$, and conclude by the preceding that the triangles BNF and CNE are similar.

It then follows that $\angle BFN = \angle CEN$, so $\angle AFN = \angle AEN$, showing that N lies indeed on the circle AEF, as desired.

Problem 3. Determine all integers (strictly) greater than 1 whose positive divisors add up to a power of 3.

Andrei Bâra

Solution. The integer 2 alone satisfies the condition in the statement. The verification offers no difficulty.

For every positive integer k, let $\sigma(k)$ denote the sum of all positive divisors of k, 1 and k inclusive. If k and ℓ are positive integers, and k is coprime to ℓ, then $\sigma(k\ell) = \sigma(k)\sigma(\ell)$.

Consider now an integer $n \geq 2$ such that $\sigma(n) = 3^N$ for some positive integer N. Let p be a prime divisor of n and let p^K be the highest power of p dividing n; that is, p^K divides n, but p^{K+1} does not. Then $1+p+\cdots+p^K = \sigma(p^K)$ divides $\sigma(n) = 3^N$, so $1+p+\cdots+p^K = 3^M$ for some positive integer M; in particular, $p \neq 3$.

Let now q be a prime divisor of $K+1$. Then p^q-1 divides $p^{K+1}-1$, so $1+p+\cdots+p^{q-1} = 3^L$ for some positive integer L. The latter forces $p^q \equiv 1 \pmod 3$.

If $q = 2$, then $p = 3^L - 1$ is even, so $p = 2$ and $2^{K+1} - 1 = 1 + 2 + \cdots + 2^K = 3^M$. The latter forces K odd. Write $\left(2^{(K+1)/2} - 1\right)\left(2^{(K+1)/2} + 1\right) = 3^M$ and notice that the two factors are relatively prime integers, to infer that $K = 1$.

If q is odd, recall that $p^q \equiv 1 \pmod 3$, so $p \equiv 1 \pmod 3$. Then $q \equiv 1 + p + \cdots + p^{q-1} \pmod 3$, and $1 + p + \cdots + p^{q-1} = 3^L$ forces $q = 3$, so $1 + p + p^2 = 3^L$. Since $p \equiv 1 \pmod 3$, it follows that $1 + p + p^2 \equiv 3 \pmod 9$, so $L = 1$ and we reach a contradiction.

Consequently, $n = 2$ is the only integer (strictly) greater than 1 whose positive divisors add up to a power of 3.

Problem 4. Let $a_0 = 1$, let $a_1 = 2$, and let $a_2 = 10$, to define $a_{k+2} = a_{k+1}^3 + a_k^2 + a_{k-1}$ for all positive integers k. Is it possible that some a_k be divisible by 2021^{2021}?

Flavian Georgescu

Solution. The answer is in the affirmative. Given any integer $N \geq 2$, we show that some a_k is divisible by N.

To this end, let $b_0 = 0$, let $b_1 = 1$, and let $b_k = a_{k-2}$ for all $k \geq 2$, so $b_{k+2} = b_{k+1}^3 + b_k^2 + b_{k-1}$ for all positive integers k.

For convenience, if a, b, c, a', b', c' are integers, write $(a, b, c) \equiv (a', b', c') \pmod{N}$ whenever $a \equiv a' \pmod{N}$, $b \equiv b' \pmod{N}$ and $c \equiv c' \pmod{N}$.

Consider now all triples (b_k, b_{k+1}, b_{k+2}), $k = 0, 1, 2, \ldots$. Since there are only finitely many (positive) residues modulo N, it follows that $(b_n, b_{n+1}, b_{n+2}) \equiv (b_m, b_{m+1}, b_{m+2}) \pmod{N}$ for some indices $n > m$. Then $b_{n-1} \equiv b_{n+2} - b_{n+1}^3 - b_n^2 \equiv b_{m+2} - b_{m+1}^3 - b_m^2 \equiv b_{m-1} \pmod{N}$, so $(b_{n-1}, b_n, b_{n+1}) \equiv (b_{m-1}, b_m, b_{m+1}) \pmod{N}$, and so on and so forth, all the way down to $(b_{n-m}, b_{n-m+1}, b_{n-m+2}) \equiv (b_0, b_1, b_2) \equiv (0, 1, 1) \pmod{N}$. Consequently, b_{n-m} is divisible by N. Finally, since $N \geq 2$, $b_0 = 0$ and $b_1 = 1$, it follows that $n - m \geq 2$, so $a_{n-m-2} = b_{n-m}$ is indeed divisible by N. This ends the proof.