Problema săptămânii 274

Fie $a, b, c > 0$. Demonstraţi că

$$\frac{a^2+b^2}{a}+\frac{b^2+c^2}{b}+\frac{c^2+a^2}{c} \geq 4\left(\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2}\right).$$

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Soluţia 1: (a autorului)
Inegalitatea se mai poate scrie succesiv astfel:

$$\sum_{cyc}\frac{b^2}{a}+a+b+c \geq 4\sum_{cyc}\frac{a^3+ab^2-ab^2}{a^2+b^2},$$

$$\sum_{cyc}\frac{b^2}{a}-(a+b+c) \geq 2(a+b+c)-4\sum_{cyc}\frac{ab^2}{a^2+b^2},$$

$$\sum_{cyc}\frac{(a-b)^2}{a} \geq 2\sum_{cyc}\left[b-\frac{2ab^2}{a^2+b^2}\right],$$

$$\sum_{cyc}\frac{(a-b)^2}{a} \geq \sum_{cyc}\frac{2b(a-b)^2}{a^2+b^2},$$

inegalitate adevărată deoarece $a^2+b^2 \geq 2ab$.

Soluţia 2: (*Marian Cucoaneş. Radu Stoleriu*)
Inegalitatea se scrie echivalent

$$\frac{a^2+b^2}{a}+\frac{b^2+c^2}{b}+\frac{c^2+a^2}{c}+\frac{4ab^2}{a^2+b^2}+\frac{4bc^2}{b^2+c^2}+\frac{4ca^2}{c^2+a^2} \geq 4a+4b+4c.$$

Ea rezultă din adunarea inegalităţii $\dfrac{a^2+b^2}{a}+\dfrac{4ab^2}{a^2+b^2} \geq 4b$ cu analoagele ei.
Inegalitatea precedentă rezultă imediat din inegalitatea mediilor, egalitatea având loc atunci când $a^2+b^2=2ab$, adică $a=b$. În concluzie, în inegalitatea din enunţ, egalitatea are loc dacă $a=b=c$.

Am primit soluţii de la: *David Ghibu, Radu Şerban, Marian Cucoaneş, Radu Stoleriu* şi *Emanuel Mazăre.*

Problem of the week no. 274

If a, b, c are positive real numbers, prove that

$$\frac{a^2+b^2}{a}+\frac{b^2+c^2}{b}+\frac{c^2+a^2}{c} \geq 4\left(\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2}\right).$$

Marius Stănean

Solution: (by the author)
The inequality can be written successively:

$$\sum_{cyc}\frac{b^2}{a}+a+b+c \geq 4\sum_{cyc}\frac{a^3+ab^2-ab^2}{a^2+b^2},$$

$$\sum_{cyc}\frac{b^2}{a}-(a+b+c) \geq 2(a+b+c)-4\sum_{cyc}\frac{ab^2}{a^2+b^2},$$

$$\sum_{cyc}\frac{(a-b)^2}{a} \geq 2\sum_{cyc}\left[b-\frac{2ab^2}{a^2+b^2}\right],$$

$$\sum_{cyc}\frac{(a-b)^2}{a} \geq \sum_{cyc}\frac{2b(a-b)^2}{a^2+b^2},$$

which is true because $a^2+b^2 \geq 2ab$.

Solution 2: (*Marian Cucoaneş. Radu Stoleriu*)
The inequality can be written equivalently

$$\frac{a^2+b^2}{a}+\frac{b^2+c^2}{b}+\frac{c^2+a^2}{c}+\frac{4ab^2}{a^2+b^2}+\frac{4bc^2}{b^2+c^2}+\frac{4ca^2}{c^2+a^2} \geq 4a+4b+4c.$$

It follows from adding $\dfrac{a^2+b^2}{a}+\dfrac{4ab^2}{a^2+b^2} \geq 4b$ with its analogues.
The previous inequality follows from the AM-GM inequality. Equality holds when $a^2+b^2=2ab$, i.e. when $a=b$. In conclusion, in the given inequality, equality holds when $a=b=c$.