Problema săptămânii 267

Notăm cu $[a, b]$ cel mai mic multiplu comun al numerelor naturale nenule a şi b.
Fie n un număr natural nenul cu proprietatea că

$$[n, n+1] > [n, n+2] > \ldots > [n, n+9].$$

Demonstraţi că $[n, n+9] > [n, n+10]$.

Concursul Arany Dániel, 2020

Soluţie: Deoarece $[a, b] = \dfrac{ab}{(a, b)}$ şi avem $n(n+1) < n(n+2) < \ldots < n(n+9)$, deducem că $(n, n+1) < (n, n+2) < \ldots < (n, n+9)$. Dar orice divizor comun al lui n şi $n+9$ este şi divizor al lui 9, deci este cel mult 9. Aşadar avem $1 \leq (n, n+1) < n(n+2) < \ldots < n(n+9) \leq 9$, prin urmare trebuie să avem $(n, n+k) = k$ pentru orice $k \in \{1, 2, \ldots, 9\}$. Deducem de aici că:

- $[n, n+9] = \dfrac{n(n+9)}{(n, n+9)} = \dfrac{n(n+9)}{9} = n + \dfrac{n^2}{9}$;
- $(n, n+2) = 2$ şi $(n, n+5) = 5$ implică $2 \mid n$, respectiv $5 \mid n$, deci $10 \mid n$, de unde se obţine imediat că $(n, n+10) = 10$, deci $[n, n+10] = \dfrac{n(n+10)}{(n, n+10)} = \dfrac{n(n+10)}{10} = n + \dfrac{n^2}{10}$.

Acum este clar că $[n, n+9] = n + \dfrac{n^2}{9} > n + \dfrac{n^2}{10} = [n, n+10]$.

Remarcă: Există o infinitate de numere n cu proprietatea din enunţ. Orice număr divizibil cu $1, 2, 3, \ldots, 9$, adică cu $[1, 2, 3, \ldots, 9] = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$ are proprietatea din enunţ. Într-adevăr, este suficient să avem $(n, n+k) = k$ pentru orice $k \in \{1, 2, \ldots, 9\}$ pentru ca $[n, n+1] > [n, n+2] > \ldots > [n, n+9]$:
$[n, n+j] > [n, n+j+1] \Leftrightarrow \dfrac{n(n+j)}{j} > \dfrac{n(n+j+1)}{j+1}$ care este evidentă.

Am primit soluţii de la *Emanuel Mazăre, Ştefan Gobej, Mihai Miculiţa* şi *David Ghibu.*

Problem of the week no. 267

Denote by $\mathrm{lcm}(a, b)$ the least common multiple of positive integers a and b.
Let n be a positive integer such that

$$\mathrm{lcm}(n, n+1) > \mathrm{lcm}(n, n+2) > \ldots > \mathrm{lcm}(n, n+9).$$

Prove that $\mathrm{lcm}(n, n+9) > \mathrm{lcm}(n, n+10)$.

Arany Dániel Contest, 2020

Solution: As $\text{lcm}(a,b) = \dfrac{ab}{\gcd(a,b)}$ and $n(n+1) < n(n+2) < \ldots < n(n+9)$, it follows that $\gcd(n, n+1) < \gcd(n, n+2) < \ldots < \gcd(n, n+9)$. But any common divisor of n and $n+9$ is also a divisor of 9, therefore it is at most 9. Thus, $1 \le \gcd(n, n+1) < \gcd n(n+2) < \ldots < \gcd(n, n+9) \le 9$, which means that we must have $\gcd(n, n+k) = k$ for all $k \in \{1, 2, \ldots, 9\}$. We conclude that:

- $\text{lcm}(n, n+9) = \dfrac{n(n+9)}{\gcd(n, n+9)} = \dfrac{n(n+9)}{9} = n + \dfrac{n^2}{9}$;
- $\gcd(n, n+2) = 2$ and $(n, n+5) = 5$ imply $2 \mid n$ and $5 \mid n$, hence $10 \mid n$, which immediately leads to $\gcd(n, n+10) = 10$ and $\text{lcm}(n, n+10) = \dfrac{n(n+10)}{\gcd(n, n+10)} = \dfrac{n(n+10)}{10} = n + \dfrac{n^2}{10}$.

Now it is clear that $\text{lcm}(n, n+9) = n + \dfrac{n^2}{9} > n + \dfrac{n^2}{10} = \text{lcm}(n, n+10)$.

Remark: There are infinitely many positive integers n with the given property. Any positive integer divisible by $1, 2, 3, \ldots, 9$, i.e. by $[1, 2, 3, \ldots, 9] = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$ satisfies the given condition. It is sufficient to have $\gcd(n, n+k) = k$ for all $k \in \{1, 2, \ldots, 9\}$ in order for $\text{lcm}(n, n+1) > \text{lcm}(n, n+2) > \ldots > \text{lcm}(n, n+9)$ to hold.

Indeed, $\text{lcm}(n, n+j) > \text{lcm}(n, n+j+1) \Leftrightarrow \dfrac{n(n+j)}{j} > \dfrac{n(n+j+1)}{j+1}$, which is obvious.