Problema săptămânii 262

Fie $a, b, c > 0$ cu $abc = 1$. Arătaţi că

$$\frac{a+3}{(a+1)^2} + \frac{b+3}{(b+1)^2} + \frac{c+3}{(c+1)^2} \geq 3.$$

Claudiu Mîndrilă

Soluţia 1:

Lemă. Fie $x,\ y > 0$. Atunci $\dfrac{1}{(1+x)^2} + \dfrac{1}{(1+y)^2} \geq \dfrac{1}{1+xy}$.

Demonstraţia lemei. Cu inegalitatea Cauchy-Buniakovski-Schwarz avem:

$(1+xy)\left(1+\dfrac{x}{y}\right) \geq (1+x)^2 \Longrightarrow \dfrac{1}{(1+x)^2} \geq \dfrac{y}{x+y} \cdot \dfrac{1}{1+xy}$ şi analog

$\dfrac{1}{(1+y)^2} \geq \dfrac{x}{x+y} \cdot \dfrac{1}{1+xy}$. Adunând ultimele 2 inegalităţi avem cerinţa.

Revenind la problemă, avem:

$\displaystyle\sum \frac{1}{(1+a)^2} = \frac{1}{2}\sum\left(\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2}\right) \geq \frac{1}{2}\sum \frac{1}{1+ab} = \frac{1}{2}\sum \frac{c}{c+1}$.

Atunci $\displaystyle\sum\frac{2}{(a+1)^2} + \sum\frac{1}{a+1} \geq \sum \frac{a+1}{a+1} = 3$ sau $\displaystyle\sum\frac{a+3}{(a+1)^2} \geq 3$ c.c.t.d.

Soluţia 2:

Eliminând numitorii, reducând termenii asemenea şi înlocuind abc cu 1 în fiecare termen care conţine abc se ajunge la inegalitatea

$$\sum_{sym} a^2b + 3\sum a^2 \geq 9 + \sum a + \sum ab.$$

Aceasta rezultă uşor din $\displaystyle\sum_{sym} a^2b \geq 6abc = 6$ (inegalitatea mediilor), $\sum a^2 \geq \sum ab$, $a^2+b^2+c^2 \geq \dfrac{1}{3}(a+b+c)^2 \geq a+b+c$ (deoarece $a+b+c \geq 3\sqrt[3]{abc} = 3$) şi $a^2+b^2+c^2 \geq a+b+c \geq 3$.

Egalitatea are loc dacă $a = b = c = 1$.

Remarcă: După cum reiese din rezolvarea (postată separat) trimisă de *Tashi Diaconescu*, inegalitatea rămâne adevărată şi dacă $a, b, c > 0$ satisfac $abc < 1$.

Remarcă: Funcţia $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \dfrac{x+3}{(x+1)^2}$ este convexă dar descrescătoare, deci $f(a) + f(b) + f(c) \geq 3f\left(\dfrac{a+b+c}{3}\right) \leq 3f\left(\sqrt[3]{abc}\right) = 3f(1) = 3$, deci aşa nu merge.

Am primit soluţii de la: *David Ghibu, Corneliu Mănescu-Avram, Ovidiu Lazăr, Ana Boiangiu, Emanuel Mazăre, Ştefan Gobej* şi *Tashi Diaconescu.*

Problem of the week no. 262

Let $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{a+3}{(a+1)^2} + \frac{b+3}{(b+1)^2} + \frac{c+3}{(c+1)^2} \geq 3.$$

Claudiu Mîndrilă

Solution 1:
Lemma. Let $x,\ y > 0$. Then $\dfrac{1}{(1+x)^2} + \dfrac{1}{(1+y)^2} \geq \dfrac{1}{1+xy}$.
Proof of the lemma. From Cauchy-Buniakovski-Schwarz inequality we have $(1+xy)\left(1+\dfrac{x}{y}\right) \geq (1+x)^2 \Longrightarrow \dfrac{1}{(1+x)^2} \geq \dfrac{y}{x+y} \cdot \dfrac{1}{1+xy}$ and, similarly, $\dfrac{1}{(1+y)^2} \geq \dfrac{x}{x+y} \cdot \dfrac{1}{1+xy}$. Adding the last two inequalities we get the one in the lemma.
Returning to the problem, we have

$$\sum \frac{1}{(1+a)^2} = \frac{1}{2}\sum\left(\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2}\right) \geq \frac{1}{2}\sum \frac{1}{1+ab} = \frac{1}{2}\sum \frac{c}{c+1}.$$

Then $\sum \dfrac{2}{(a+1)^2} + \sum \dfrac{1}{a+1} \geq \sum \dfrac{a+1}{a+1} = 3$ i.e. $\sum \dfrac{a+3}{(a+1)^2} \geq 3$.

Solution 2:
Clearing denominators, canceling similar terms and replacing abc by 1 in each term that contains abc reduces the given inequality to

$$\sum_{sym} a^2 b + 3\sum a^2 \geq 9 + \sum a + \sum ab.$$

This last inequality follows easily from: $\sum\limits_{sym} a^2 b \geq 6abc = 6$ (AM-GM inequality), $\sum a^2 \geq \sum ab$, $a^2+b^2+c^2 \geq \dfrac{1}{3}(a+b+c)^2 \geq a+b+c$ (because $a+b+c \geq 3\sqrt[3]{abc} = 3$) and $a^2+b^2+c^2 \geq a+b+c \geq 3$.
Equality holds if and only if $a = b = c = 1$.