Problem 1. Let n $(n \geq 1)$ be an integer. Consider the equation

$$2 \cdot \left\lfloor \frac{1}{2x} \right\rfloor - n + 1 = (n+1)(1 - nx),$$

where x is the unknown real variable.
(a) Solve the equation for $n = 8$.
(b) Prove that there exists an integer n for which the equation has at least 2021 solutions.
(For any real number y by $\lfloor y \rfloor$ we denote the largest integer m such that $m \leq y$.)

Solution. Let $k = \left[\frac{1}{2x}\right], k \in \mathbb{Z}$.
(a) For $n = 8$, the equation becomes

$$k = \left[\frac{1}{2x}\right] = 8 - 36x \Rightarrow x \neq 0 \text{ and } x = \frac{8-k}{36}.$$

Since $x \neq 0$, we have $k \neq 8$, and the last relation implies $k = \left[\frac{1}{2x}\right] = \left[\frac{18}{8-k}\right]$. Checking signs, we see that $0 < k < 8$. By direct verification, we find the solutions $k = 3$ (hence $x = \frac{5}{36}$ and $k = 4$ (hence $x = \frac{1}{9}$).

(b) From the given equation we have $x \neq 0$ and $x = \frac{2(n-k)}{n(n+1)}$. Therefore, $k \neq n$ and $k = \left[\frac{1}{2x}\right] = \left[\frac{n(n+1)}{4(n-k)}\right]$. Again, checking signs we see that $0 \leq k < n$. The last equation implies

$$k \leq \frac{n(n+1)}{4(n-k)} < k+1 \Rightarrow \begin{cases} (2k-n)^2 + n \geq 0 \\ (2k+1-n)^2 < n+1 \end{cases} \Rightarrow$$

$$\Rightarrow \frac{n-1-\sqrt{n+1}}{2} < k < \frac{n-1+\sqrt{n+1}}{2} \qquad (2)$$

Conversely, if $k \in \mathbb{Z}$ satisfies (2) and $0 < k < n$, then $x = \frac{2(n-k)}{n(n+1}$ is a solution to the given equation. It remains to note that choosing n such that $\sqrt{n+1} > 2021$ ensures that there exist at least 2021 integer values of k which satisfy (2).

Problem 2. For any set $A = \{x_1, x_2, x_3, x_4, x_5\}$ of five distinct positive integers denote by S_A the sum of its elements, and denote by T_A the number of triples (i, j, k) with $1 \leqslant i < j < k \leqslant 5$ for which $x_i + x_j + x_k$ divides S_A.

Find the largest possible value of T_A.

Solution. We will prove that the maximum value that T_A can attain is 4. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five positive integers such that $x_1 < x_2 < x_3 < x_4 < x_5$. Call a triple (i, j, k) with $1 \leqslant i < j < k \leqslant 5$ *good* if $x_i + x_j + x_k$ divides S_A. None of the triples $(3, 4, 5), (2, 4, 5), (1, 4, 5), (2, 3, 5), (1, 3, 5)$ is good, since, for example

$$x_5 + x_3 + x_1 \mid S_A \Rightarrow x_5 + x_3 + x_1 \mid x_2 + x_4$$

which is impossible since $x_5 > x_4$ and $x_3 > x_2$. Analogously we can show that any triple of form $(x, y, 5)$ where $y > 2$ isn't good.

By above, the number of good triples can be at most 5 and only triples (1,2,5), (2,3,4), (1,3,4), (1,2,4), (1,2,3) can be good. But if triples (1,2,5) and (2,3,4) are simultaneously good we have that:

$$x_1 + x_2 + x_5 \mid x_3 + x_4 \Rightarrow x_5 < x_3 + x_4 \tag{1}$$

and

$$x_2 + x_3 + x_4 \mid x_1 + x_5 \Rightarrow x_2 + x_3 + x_4 \leqslant x_1 + x_5 \overset{(1)}{<} x_1 + x_3 + x_4 < x_2 + x_3 + x_4,$$

which is impossible. Therefore, $T_A \leqslant 4$.

Alternatively, one can prove the statement above by adding up the two inequalities $x_1 + x_2 + x_4 < x_3 + x_4$ and $x_2 + x_3 + x_4 < x_1 + x_5$ that are derived from the divisibilities.

To show that $T_A = 4$ is possible, consider the numbers $1, 2, 3, 4, 494$. This works because $6 \mid 498$, $7 \mid 497$, $8 \mid 496$, and $9 \mid 495$.□

Remark. The motivation for construction is to realize that if we choose x_1, x_2, x_3, x_4 we can get all the conditions x_5 must satisfy. Let $S = x_1 + x_2 + x_3 + x_4$. Now we have to choose x_5 such that

$$S - x_i \mid x_i + x_5, \text{ i.e. } x_5 \equiv -x_i \mod (S - x_i) \forall i \in \{1, 2, 3, 4\}.$$

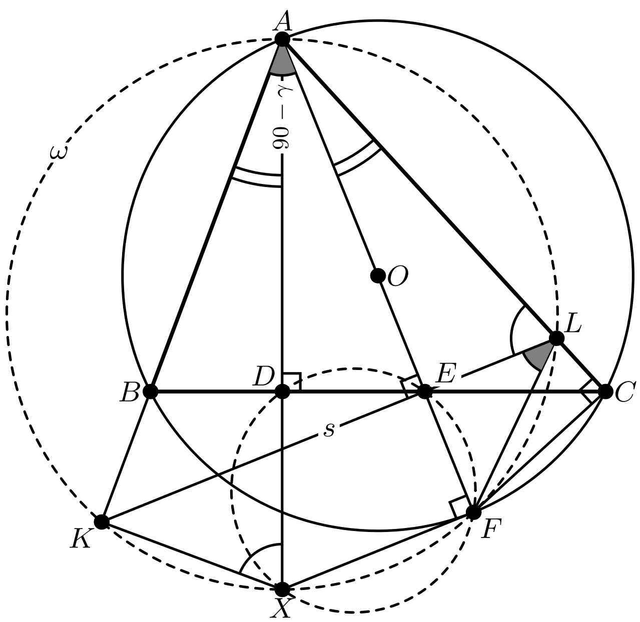
By the Chinese Remainder Theorem it is obvious that if $S - x_1, S - x_2, S - x_3, S - x_4$ are pairwise coprime, such x_5 must exist. To make all these numbers pairwise coprime it's natural to take x_1, x_2, x_3, x_4 to be all odd and then solve mod 3 issues. Fortunately it can be seen that $1, 5, 7, 11$ easily works because $13, 17, 19, 23$ are pairwise coprime.

However, even without the knowledge of this theorem it makes sense intuitively that this system must have a solution for some x_1, x_2, x_3, x_4. By taking $(x_1, x_2, x_3, x_4) = (1, 2, 3, 4)$ we get pretty simple system which can be solved by hand rather easily.

Problem 3. Let ABC be an acute scalene triangle with circumcenter O. Let D be the foot of the altitude from A to the side BC. The lines BC and AO intersect at E. Let s be the line through E perpendicular to AO. The line s intersects AB and AC at K and L, respectively. Denote by ω the circumcircle of triangle AKL. Line AD intersects ω again at X.

Prove that ω and the circumcircles of triangles ABC and DEX have a common point.

Solution.



Let us denote angles of triangle ABC with α, β, γ in a standard way. By basic angle-chasing we have

$$\angle BAD = 90^\circ - \beta = \angle OAC \text{ and } \angle CAD = \angle BAO = 90^\circ - \gamma.$$

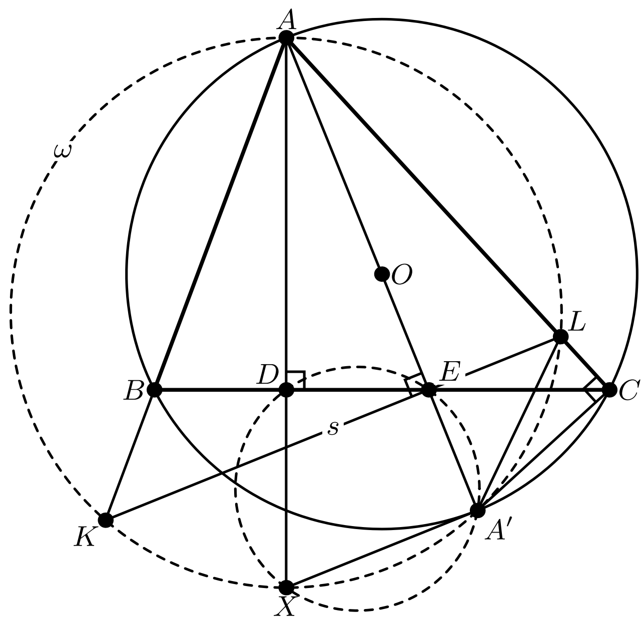
Using the fact that lines AE and AX are isogonal with respect to $\angle KAL$ we can conclude that X is an A-antipode on ω. (This fact can be purely angle-chased: we have

$$\angle KAX + \angle AXK = \angle KAX + \angle ALK = 90^\circ - \beta + \beta = 90^\circ$$

which implies $\angle AKX = 90^\circ$). Now let F be the projection of X on the line AE. Using that AX is a diameter of ω and $\angle EDX = 90^\circ$ it's clear that F is the intersection point of ω and the circumcircle of triangle DEX. Now it suffices to show that $ABFC$ is cyclic. We have $\angle KLF = \angle KAF = 90^\circ - \gamma$ and from $\angle FEL = 90^\circ$ we have that $\angle EFL = \gamma = \angle ECL$ so quadrilateral $EFCL$ is cyclic. Next, we have

$$\angle AFC = \angle EFC = 180^\circ - \angle ELC = \angle ELA = \beta$$

(where last equality holds because of $\angle AEL = 90^\circ$ and $\angle EAL = 90^\circ - \beta$).$\square$



Solution 2. We have $\angle BAD = 90^\circ - \beta = \angle OAC$ and that AX is the diameter of ω. Also we note that

$$\angle ALK = \beta,\ \angle KLC = 180^\circ - \beta = \angle KBC$$

so $BKCL$ is cyclic. Let AO intersect circumcircle of ABC again at A'. We will show that A' is the desired concurrence point. Obviously AA' is the diameter of circumcircle of triangle ABC so $\angle A'CA = 90^\circ$ which implies that $A'CLE$ is cyclic. From power of point E we have that $EK \cdot EL = EB \cdot EC = EA \cdot EA'$ so we can conclude that $A' \in \omega$. Now using the fact that AX is a diameter of ω implies $\angle AXA' = 90^\circ$ we have that $DXA'E$ is cyclic because of $\angle EDX = 90^\circ$ which finishes the proof.$\square$

Problem 4. Let M be a subset of the set of 2021 integers $\{1, 2, 3, \ldots, 2021\}$ such that for any three elements (not necessarily distinct) a, b, c of M we have $|a + b - c| > 10$. Determine the largest possible number of elements of M.

Solution. The set $M = \{1016, 1017, \ldots, 2021\}$ has 1006 elements and satisfies the required property, since $a, b, c \in M$ implies that $a + b - c \geqslant 1016 + 1016 - 2021 = 11$. We will show that this is optimal.

Suppose M satisfies the condition in the problem. Let k be the minimal element of M. Then $k = |k + k - k| > 10 \Rightarrow k \geqslant 11$. Note also that for every m, the integers $m, m + k - 10$ cannot both belong to M, since $k + m - (m + k - 10) = 10$.

Claim 1: M contains at most $k - 10$ out of any $2k - 20$ consecutive integers.
Proof: We can partition the set $\{m, m+1, \ldots, m+2k-21\}$ into $k-10$ pairs as follows:

$$\{m, m + k - 10\}, \{m + 1, m + k - 9\}, \ldots, \{m + k - 11, m + 2k - 21\},$$

It remains to note that M can contain at most one element of each pair.

Claim 2: M contains at most $[(t + k - 10)/2]$ out of any t consecutive integers.
Proof: Write $t = q(2k - 20) + r$ with $r \in \{0, 1, 2 \ldots, 2k - 21\}$. From the set of the first $q(2k - 20)$ integers, by Claim 1 at most $q(k - 10)$ can belong to M. Also by claim 1, it follows that from the last r integers, at most $\min\{r, k - 10\}$ can belong to M.

Thus,

- If $r \leqslant k - 10$, then at most

$$q(k - 10) + r = \frac{t + r}{2} \leqslant \frac{t + k - 10}{2} \text{ integers belong to } M.$$

- If $r > k - 10$, then at most

$$q(k - 10) + k - 10 = \frac{t - r + 2(k - 10)}{2} \leqslant \frac{t + k - 10}{2} \text{ integers belong to } M.$$

By Claim 2, the number of elements of M amongst $k + 1, k + 2, \ldots, 2021$ is at most

$$\left[\frac{(2021 - k) + (k - 10)}{2}\right] = 1005.$$

Since amongst $\{1, 2, \ldots, k\}$ only k belongs to M, we conclude that M has at most 1006 elements as claimed. $\square$