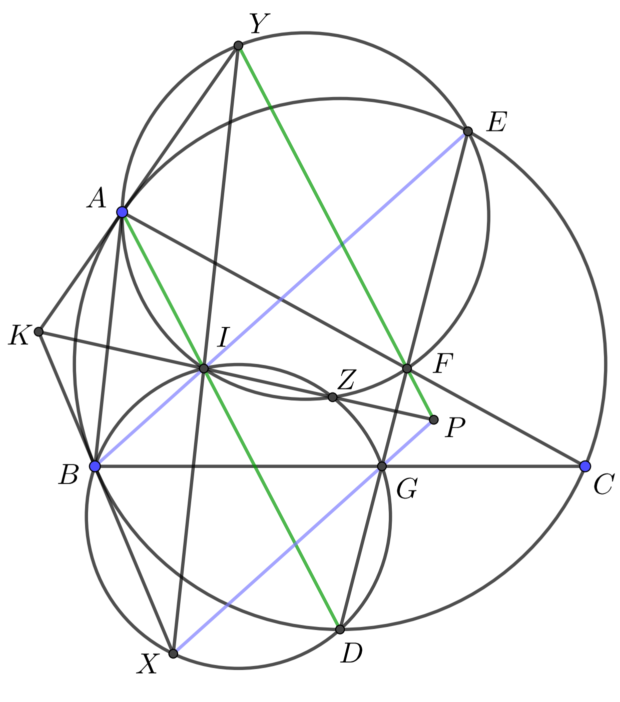
Problema săptămânii 245

Fie ABC un triunghi, I centrul cercului său înscris, iar ω cercul său circumscris. Fie D şi E al doilea punct de intersecţie dintre ω şi AI, respectiv BI. Dreapta DE intersectează AC în F şi BC în G. Fie P punctul de intersecţie dintre paralela prin F la AD şi paralela prin G la BE. Tangentele în A şi B la ω se intersectează în K. Arătaţi că dreptele AE, BD şi KP sunt paralele sau concurente.

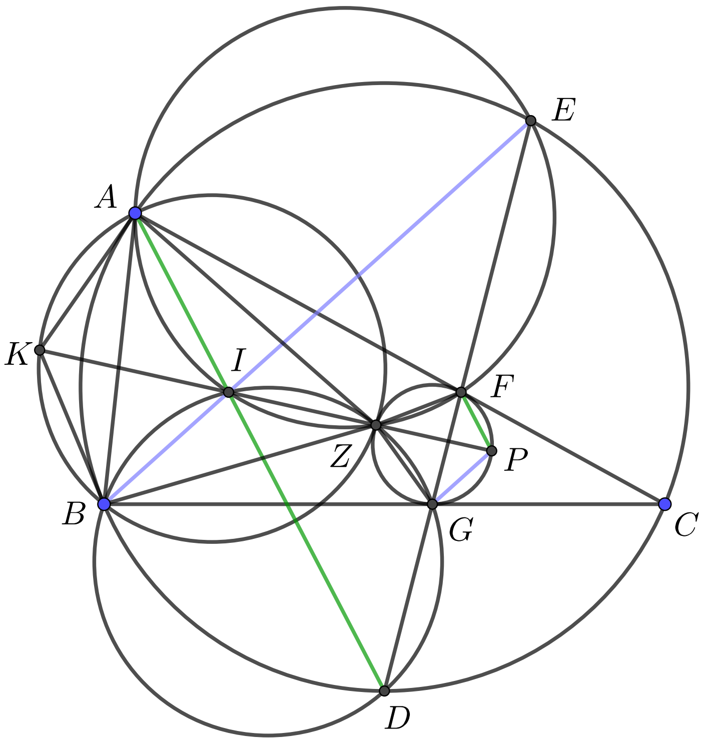
Irena Majcen şi Kris Stopar, Shl IMO, 2011

Soluţia 1: Avem $m(\sphericalangle BGD) = \frac{1}{2}\big(m(\overset{\frown}{BD}) + m(\overset{\frown}{CE})\big) = \frac{1}{2}\big(m(\sphericalangle A) + m(\sphericalangle B)\big) = m(\sphericalangle BID)$, deci patrulaterul $BIGD$ este inscriptibil. Analog, $AIFE$ este inscriptibil. Fie ω_1 şi ω_2 cercurile circumscrise patrulaterelor $BIGD$, respectiv $AIFE$ şi $\omega_1 \cap \omega_2 = \{I, Z\}$. Fie $\{B, X\} = KB \cap \omega_1$ şi $\{A, Y\} = KA \cap \omega_2$. Atunci $\sphericalangle KAB \equiv \sphericalangle KBA \equiv \sphericalangle C$ şi, pe de altă parte, $\sphericalangle AYI \equiv \sphericalangle AEI \equiv \sphericalangle AEB \equiv \sphericalangle KAB$, deci $YI \parallel AB$. Analog se arată că $XI \parallel AB$, deci punctele X, I, Y sunt coliniare. Triunghiurile KAB şi KXY sunt isoscele, deci $KA \cdot KY = KB \cdot KX$, prin urmare K se află pe axa radicală, ZI, a cercurilor ω_1 şi ω_2.

Totodată, $m(\sphericalangle BXG) = m(\sphericalangle BDG) = m(\sphericalangle BDE) = m(\sphericalangle C) + \frac{1}{2}m(\sphericalangle B)$, iar $m(\sphericalangle XBG) = m(\sphericalangle A)$, deci, din triunghiul XBG rezultă că $m(\sphericalangle BGX) = \frac{1}{2}m(\sphericalangle B) = m(\sphericalangle IBG)$, deci că $XG \parallel BI$. Rezultă că $P \in GX$ şi, analog, $P \in FY$. În fine, $\sphericalangle EFY \equiv \sphericalangle EIY \equiv \sphericalangle BIX \equiv \sphericalangle BGX \equiv \sphericalangle IBG \equiv \sphericalangle IXG$, deci patrulaterul $XYFG$ este inscriptibil. Atunci $PF \cdot PY = PG \cdot PX$, deci P are aceeaşi putere faţă de cercurile ω_1 şi ω_2, prin urmare $P \in ZI$. Aşadar punctele K, I, X, P sunt coliniare. Atunci BD este axa radicală a cercurilor ω şi ω_1, AE este axa radicală a cercurilor ω şi ω_2, iar KP este axa radicală a cercurilor ω_1 şi ω_2. Cele trei axe radicale sunt fie paralele, fie concurente în centrul radical al celor trei cercuri.



Soluţia 2: (*Ana Duguleanu*)
Ca mai sus se arată că patrulaterele $AIFE$ şi $BIGD$ sunt inscriptibile.
Avem $\sphericalangle IZC \equiv \sphericalangle IDB \equiv \sphericalangle C$ şi, analog, $\sphericalangle AZI \equiv \sphericalangle C$. De asemenea, $\sphericalangle KAB \equiv \sphericalangle KBA \equiv \sphericalangle C$, deci $m(\sphericalangle AKB) = 180^\circ - 2m(\sphericalangle C) = 180^\circ - m(\sphericalangle AZB)$, deci patrulaterul $AKBZ$ este inscriptibil.
Atunci $\sphericalangle KZB \equiv \sphericalangle KAB \equiv \sphericalangle IZB$ arată că $K \in IZ$.
Pe de altă parte, $m(\sphericalangle FZG) = 360^\circ - m(\sphericalangle IZF) - m(\sphericalangle IZG) = 180^\circ - m(\sphericalangle IZF) + 180^\circ - m(\sphericalangle IZG) = m(\sphericalangle IED) + m(\sphericalangle IDE) = \frac{1}{2}m(\sphericalangle A) + \frac{1}{2}m(\sphericalangle B) = m(\sphericalangle GFP) + m(\sphericalangle FGP) = 180^\circ - m(\sphericalangle FPG)$, deci patrulaterul $FZGP$ este inscriptibil.
Atunci $m(\sphericalangle FZP) = m(\sphericalangle FGP) = \frac{1}{2}m(\sphericalangle A) = m(\sphericalangle IEF) = 180^\circ - m(\sphericalangle IZF)$, deci $P \in ZI$.
Concluzia rezultă la fel ca la soluţia 1.



Remarcă: (*Emanuel Mazăre*)
Cele trei drepte sunt paralele dacă şi numai dacă $m(\sphericalangle C) = 60^\circ$.

Alte soluţii găsiţi pe AoPS.

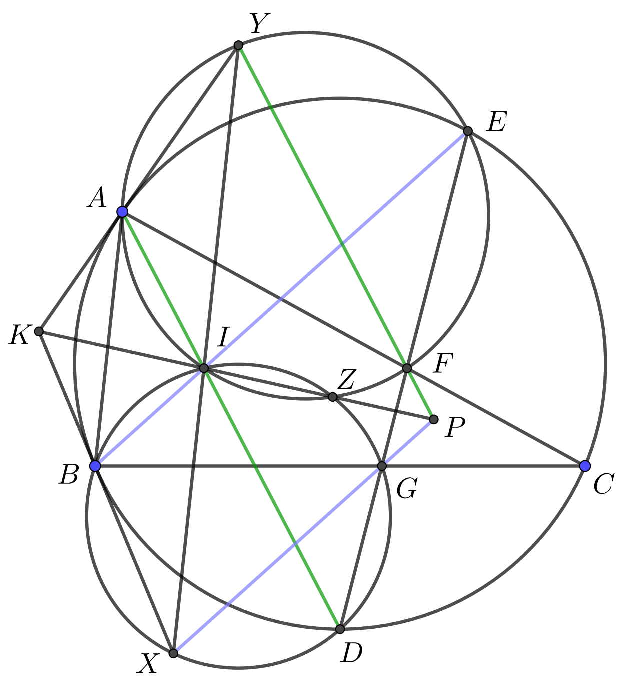
Am primit soluţii de la: *Ana Duguleanu, Cezara Danciu* şi *Emanuel Mazăre*.

Problem of the week no. 245

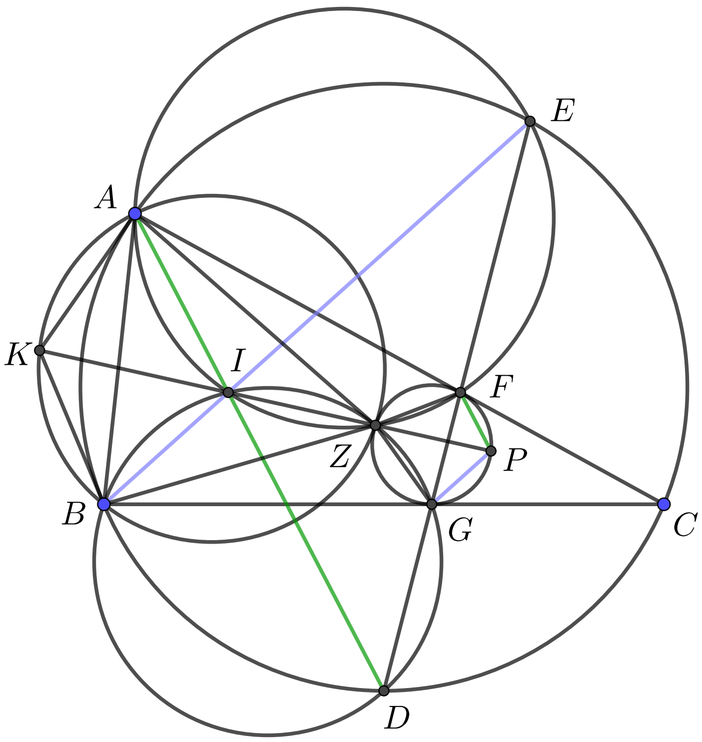
Let ABC be a triangle, I its incenter and ω its circumcircle. Let D and E be the second intersection point between ω and the lines AI and BI, respectively. Line DE cuts AC at F and BC at G. The parallel through F to AD meets the parallel through G to BF at P. The tangent lines at A and B to ω meet at K. Prove that lines AE, BD, and KP are either parallel, or coincide.

Irena Majcen and Kris Stopar, IMO ShL, 2011

Solution 1: As $\sphericalangle BGD = \frac{1}{2}\big(\widehat{BD} + \widehat{CE}\big) = \frac{1}{2}\big(m(\sphericalangle A) + m(\sphericalangle B)\big) = m(\sphericalangle BID)$, quadrilateral $BIGD$ is cyclic. Similarly, $AIFE$ is also cyclic. Let ω_1 and ω_2 be the circumcircles of $BIGD$ and $AIFE$. Let $\omega_1 \cap \omega_2 = \{I, Z\}$, $\{B, X\} = KB \cap \omega_1$ and $\{A, Y\} = KA \cap \omega_2$. Then $\sphericalangle KAB \equiv \sphericalangle KBA \equiv \sphericalangle C$ and, on the other hand, $\sphericalangle AYI \equiv \sphericalangle AEI \equiv \sphericalangle AEB \equiv \sphericalangle KAB$, i.e. $YI \parallel AB$. Similarly, $XI \parallel AB$, i.e. X, I, Y are collinear. Triangles KAB and KXY are isosceles, therefore $KA \cdot KY = KB \cdot KX$, which means that K lies on the radical axis, ZI, of the circles ω_1 and ω_2.
Moreover, $\sphericalangle BXG = \sphericalangle BDG = \sphericalangle BDE = \sphericalangle C + \frac{1}{2}\sphericalangle B$, while $\sphericalangle XBG = \sphericalangle A$, thus, from triangle XBG we get $\sphericalangle BGX = \frac{1}{2}\sphericalangle B = \sphericalangle IBG$, i.e. $XG \parallel BI$. It follows that $P \in GX$ and, similarly, $P \in FY$.
Finally, $\sphericalangle EFY \equiv \sphericalangle EIY \equiv \sphericalangle BIX \equiv \sphericalangle BGX \equiv \sphericalangle IBG \equiv \sphericalangle IXG$, which means that $XYFG$ is cyclic. Then $PF \cdot PY = PG \cdot PX$, showing that P has the same power w.r.t. circles ω_1 and ω_2, therefore $P \in ZI$. We conclude that points K, I, X, P are collinear.
We have: BD is the radical axis of circles ω and ω_1, AE is the radical axis of the circles ω and ω_2, and KP is the radical axis of circles ω_1 and ω_2. These three radical axes are either parallel or they concur in the radical center of the three circles.



Solution 2: (*Ana Duguleanu*)
As above, one shows that $AIFE$ and $BIGD$ are cyclic.
We have $\sphericalangle IZC \equiv \sphericalangle IDB \equiv \sphericalangle C$ and, similarly, $\sphericalangle AZI \equiv \sphericalangle C$. Also, $\sphericalangle KAB \equiv \sphericalangle KBA \equiv \sphericalangle C$, therefore $\sphericalangle AKB = 180° - 2\sphericalangle C = 180° - \sphericalangle AZB$, which shows that $AKBZ$ is cyclic.
It follows that $\sphericalangle KZB \equiv \sphericalangle KAB \equiv \sphericalangle IZB$, which means that $K \in IZ$.
On the other hand, $\sphericalangle FZG = 360° - \sphericalangle IZF - \sphericalangle IZG = 180° - \sphericalangle IZF + 180° - \sphericalangle IZG = \sphericalangle IED + \sphericalangle IDE = \frac{1}{2}\sphericalangle A + \frac{1}{2}\sphericalangle B = \sphericalangle GFP + \sphericalangle FGP = 180° - \sphericalangle FPG$, therefore $FZGP$ is cyclic.
We obtain that $\sphericalangle FZP = \sphericalangle FGP = \frac{1}{2}\sphericalangle A = \sphericalangle IEF = 180° - \sphericalangle IZF$, which shows that $P \in ZI$.
The conclusion follows the same way as in solution 1.



Remark: (*Emanuel Mazăre*)
The three lines are parallel if and only if $\sphericalangle C = 60°$.

Other solutions can be found on AoPS.