2021 Stars of Mahtematics, Junior Grade — Solution to Problem 1

Problem 1. For every integer $n \geq 3$, let s_n be the sum of all primes (strictly) less than n. Show that there are infinitely many integers $n \geq 3$ such that s_n is coprime to n.

Russian Competition

Solution. It is clearly sufficient to show that at least one of the entries of every pair of consecutive odd primes satisfies the condition in the statement. (Inidentally, notice that $s_5 = 2 + 3 = 5$, so 5 and s_5 are not relatively prime.)

For a prime p, the condition is that s_p be not divisible by p. Let (p, q) be a pair of consecutive odd primes; say, $p < q$, so $s_q = s_p + p$. Clearly, it is sufficient to show that, if $s_p = kp$ for some positive integer k, then $s_q = (k+1)p$ is not divisible by q. Alternatively, but equivalently, that $k + 1$ is not divisible by q.

To prove the latter, notice that $kp = s_p < 1+2+\cdots+(p-1) = \frac{1}{2}p(p-1)$, so $k < \frac{1}{2}(p-1)$. Consequently, $k + 1 < \frac{1}{2}(p + 1) < p < q$, showing that $k + 1$ is indeed not divisible by q.

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 2

Problem 2. Fix integers $m \geq 3$ and $n \geq 3$. Each cell of an array with m rows and n columns is coloured one of two colours such that:

(1) Both colours occur on every column; and
(2) On every two rows the cells on the same column share colour on exactly k columns.

Show that, if m is odd, then

$$\frac{n(m-1)}{2m} \leq k \leq \frac{n(m-2)}{m}.$$

The Problem Selection Committee

Convention. The pairs considered in both solutions in the sequel are all *unordered*. For convenience, two cells on the same column form a *vertical pair*. A *bicolour* vertical pair is one whose cells bear distinct colours; otherwise the vertical pair is *monochromatic*.

Solution 1. Count the total number of bicolour vertical pairs. There are $\frac{1}{2}m(m-1)$ pairs of rows, each of which contains exactly $n-k$ bicolour vertical pairs, so the array contains exactly $\frac{1}{2}m(m-1)(n-k)$ such pairs.

We will show that the total number of monochromatic vertical pairs in the array is at least $n(m-1)$ and, if m is odd, at most $\frac{1}{4}n(m^2-1)$; if m is even, it is at most $\frac{1}{4}nm^2$. The conclusion then follows at once, by the count in the preceding paragraph — the obvious manipulations and calculations are omitted.

To establish the bounds, it is sufficient to show that the number of bicolour pairs along any column is at least $m-1$ and, if m is odd, at most $\frac{1}{4}(m^2-1)$; if m is even, it is at most $\frac{1}{4}m^2 > \frac{1}{4}(m^2-1)$.

Fix a column and let p be the number of cells of one colour on that column. Clearly, there are $p(m-p)$ bicolour pairs along the column.

Since $(p-1)(m-p-1) \geq 0$, it follows that $p(m-p) \geq m-1$, showing that the number of bicolour pairs along the column is at least $m-1$.

On the other hand, $p(m-p) \leq \frac{1}{4}m^2$, so $p(m-p) \leq \lfloor \frac{1}{4}m^2 \rfloor$. If m is odd, then $\lfloor \frac{1}{4}m^2 \rfloor = \frac{1}{4}(m^2-1)$, so the number of bicolour pairs along the column is at most $\frac{1}{4}(m^2-1)$.

This completes the argument and concludes the proof.

Solution 2. Count the total number of monochromatic vertical pairs. There are $\frac{1}{2}m(m-1)$ pairs of rows, each of which contains exactly k monochromatic vertical pairs, so the array contains exactly $\frac{1}{2}km(m-1)$ such pairs.

We will show that the total number of monochromatic vertical pairs in the array is at most $\frac{1}{2}n(m-1)(m-2)$ and, if m is odd, at least $\frac{1}{4}n(m-1)^2$; if m is even, it is at least $\frac{1}{4}nm(m-2)$. The conclusion then follows at once, by the count in the preceding paragraph.

To establish the bounds, it is sufficient to show that the number of monochromatic pairs along any column is at most $\frac{1}{2}(m-1)(m-2)$ and, if m is odd, at least $\frac{1}{4}(m-1)^2$; if m is even, it is at least $\frac{1}{4}m(m-2) < \frac{1}{4}(m-1)^2$.

Fix a column and let p be the number of cells of one colour along that column, and let $q = m-p$ be the number of cells of the other colour. The cells along the column form $\frac{1}{2}p(p-1)$

pairs of one colour, and $\frac{1}{2}q(q-1)$ pairs of the other colour. Hence there are $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)$ monochromatic pairs along the column. It is easily seen that this number falls between the two bounds mentioned in the preceding paragraph.

To check that $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1) \le \frac{1}{2}(m-1)(m-2)$, write $q = m - p$ and carry out calculations to obtain the equivalent inequality $(p-1)(p-m+1) \le 0$. This holds, since p lies precisely in the range 1 through $m-1$.

Similarly, checking that $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1) \ge \frac{1}{4}(m-1)^2$ if m is odd, amounts to $(2p-m)^2 \ge 1$, which is clearly the case by an obvious parity argument.

The weaker inequality, $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1) = \frac{1}{2}(p^2+q^2) - \frac{1}{2}(p+q) = \frac{1}{2}(p^2+q^2) - \frac{1}{2}m \ge \frac{1}{4}(p+q)^2 - \frac{1}{2}m = \frac{1}{4}m(m-2)$, holds whatever the parity of m; in particular, if m is even.

This completes the argument and concludes the proof.

Remark. The bounds in the statement can both be achieved. For instance, let $m = 5$, $n = 10$, and write b for 'blue' and r for 'red'. The two arrays below achieve the lower bound $k = 4$ (left) and the upper bound $k = 6$ (right), respectively:

$$\begin{matrix} b & b & b & b & r & r & r & r & r & r \\ b & r & r & r & b & b & b & r & r & r \\ r & b & r & r & b & r & r & b & b & r \\ r & r & b & r & r & b & r & b & r & b \\ r & r & r & b & r & r & b & r & b & b \end{matrix} \qquad \begin{matrix} b & r & r & r & r & r & b & b & b & b \\ r & b & r & r & r & b & r & b & b & b \\ r & r & b & r & r & b & b & r & b & b \\ r & r & r & b & r & b & b & b & r & b \\ r & r & r & r & b & b & b & b & b & r \end{matrix}$$

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 3

Problem 3. Let ABC be a triangle and let M be the midpoint of the side BC. The reflexion of the line AM in the internal bisectrix of the angle $\angle BAC$ crosses the circumcircle of the triangle ABC again at D. Let Q and R be the feet of the perpendiculars from D on the lines AC and AB, respectively, and let X be a point on the line QR, different from both Q and R. The line through X and perpendicular to DX crosses the lines AC and AB at V and W, respectively. Show that the midpoint of the segment VW lies on the line BC.

Adapted from The Amer. Math. Monthly

Solution. (*by The Problem Selection Committee*) Let P be the foot of the perpendicular from D on the line BC, and recall that P, Q and R are collinear — the line ℓ through these points is the Simson line of D with respect to the triangle ABC. If $X = P$, the conclusion is clear, since $V = C$ and $W = B$.

For a generic X on ℓ, that is, $X \neq P, Q, R$, the line VW crosses the line BC at some point U. We will show that U is the midpoint of the segment VW.

Apply Menelaus' theorem to triangle AVW and transversal UBC to write

$$\frac{UV}{UW} \cdot \frac{BW}{BA} \cdot \frac{CA}{CV} = 1.$$

It is therefore sufficient to show that $CV/BW = CA/BA$. The latter is a consequence of the fact that (CDV, BDW), (ACD, AMB) and (ABD, AMC) are pairs of similar triangles. Indeed, assuming these similarities and recalling that $MB = MC$, the desired equality follows from the corresponding similarity ratios below:

$$\frac{CV}{BW} = \frac{CD}{BD}, \qquad \frac{CD}{MB} = \frac{AC}{AM}, \qquad \frac{BD}{MC} = \frac{AB}{AM}.$$

We now turn to prove the three similarities above. The last two offer no difficulty — they both follow by isogonality at A and standard angle chase in the circle through A, B, C, D. For instance, $\angle(AD, AC) = \angle(AB, AM)$, by isogonality at A, and $\angle(DC, DA) = \angle(BC, BA) = \angle(BM, BA)$, on account of A, B, C, D being concyclic. The triangles ACD and AMB are therefore similar. The pair of triangles (ABD, AMC) is dealt with similarly.

To deal with the pair (CDV, BDW), proceed by angle chase in the different cyclic quadrangles that form in the configuration. Thus, $\angle(CD, CV) = \angle(CD, CA) = \angle(BA, BD) = \angle(BW, BD)$, on account of A, B, C, D being concyclic; and $\angle(VC, VD) = \angle(VQ, VD) = \angle(XQ, XD) = \angle(XR, XD) = \angle(WR, WD) = \angle(WB, WD)$, where the third equality holds on account of D, Q, V, X being concyclic (Q and X both lie on the circle on diameter DV), and the fifth — on account of D, R, W, X being concyclic (X and R both lie on the circle on diameter DW). The triangles CDV and BDW are therefore similar. This completes the argument and concludes the proof.

Remark. The particular case where X is the orthogonal projection of D on ℓ shows that P is the midpoint of the segment QR.

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 4

Problem 4. Let k be a positive integer, and let a, b and c be positive real numbers. Show that
$$a\big(1-a^k\big)+b\big(1-(a+b)^k\big)+c\big(1-(a+b+c)^k\big)<\frac{k}{k+1}.$$

* * *

Solution. Let S denote the sum in the left-hand member of the required inequality, and let $x=a$, $y=a+b$ and $z=a+b+c$. Then $b=y-x$, $c=z-y$, so $0<x<y<z$, and $S=x\big(1-x^k\big)+(y-x)\big(1-y^k\big)+(z-y)\big(1-z^k\big)$.

The special case of the AM-GM inequality, $\frac{1}{k+1}\big(u^{k+1}+kv^{k+1}\big)\geq uv^k$, $u\geq 0$, $v\geq 0$, is used in the sequel; the inequality is strict unless $u=v$.

The particular case, $\frac{1}{k+1}u^{k+1}+\frac{k}{k+1}\geq u$, $u\geq 0$, is used in the last relation below.

The required inequality now follows from the chain of relations below:

$$\begin{aligned}
S&=x\big(1-x^k\big)+(y-x)\big(1-y^k\big)+(z-y)\big(1-z^k\big)\\
&=x+(y-x)+(z-y)-x^{k+1}-(y-x)y^k-(z-y)z^k\\
&=z-x^{k+1}+xy^k-y^{k+1}+yz^k-z^{k+1}\\
&<z-x^{k+1}+\tfrac{1}{k+1}\big(x^{k+1}+ky^{k+1}\big)-y^{k+1}+\tfrac{1}{k+1}\big(y^{k+1}+kz^{k+1}\big)-z^{k+1}\\
&=z+\Big(-x^{k+1}+\tfrac{1}{k+1}x^{k+1}\Big)+\Big(\tfrac{k}{k+1}y^{k+1}-y^{k+1}+\tfrac{1}{k+1}y^{k+1}\Big)+\Big(\tfrac{k}{k+1}z^{k+1}-z^{k+1}\Big)\\
&=z-\tfrac{k}{k+1}x^{k+1}-\tfrac{1}{k+1}z^{k+1}=\tfrac{k}{k+1}-\tfrac{k}{k+1}x^{k+1}-\Big(\tfrac{1}{k+1}z^{k+1}+\tfrac{k}{k+1}-z\Big)\leq\tfrac{k}{k+1}-\tfrac{k}{k+1}x^{k+1}\\
&<\tfrac{k}{k+1}.
\end{aligned}$$

This ends the proof.