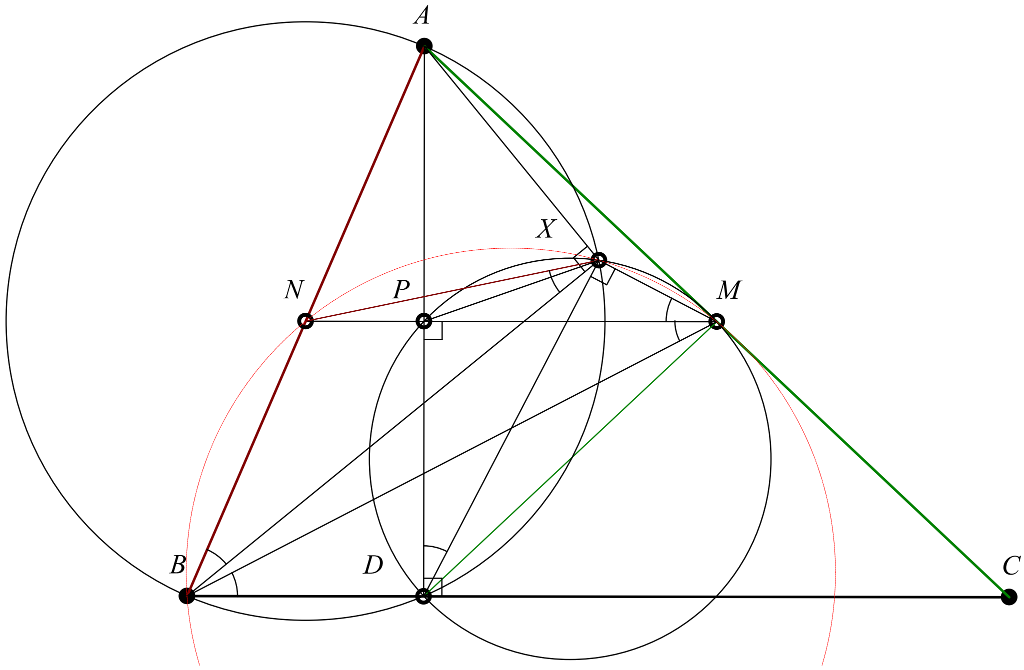
Problema săptămânii 241

Fie ABC un triunghi ascuţitunghic în care D este piciorul înălţimii duse din vârful A şi M este mijlocul laturii $[AC]$. Presupunând că există un punct X, având proprietatea că $m(\widehat{AXB}) = m(\widehat{DXM}) = 90°$ şi în plus punctele X şi C se găseasc de o parte şi de alta a dreptei BM, arătaţi că $m(\widehat{XMB}) = 2 \cdot m(\widehat{MBC})$.

antrenament EGMO, Italia, 2021

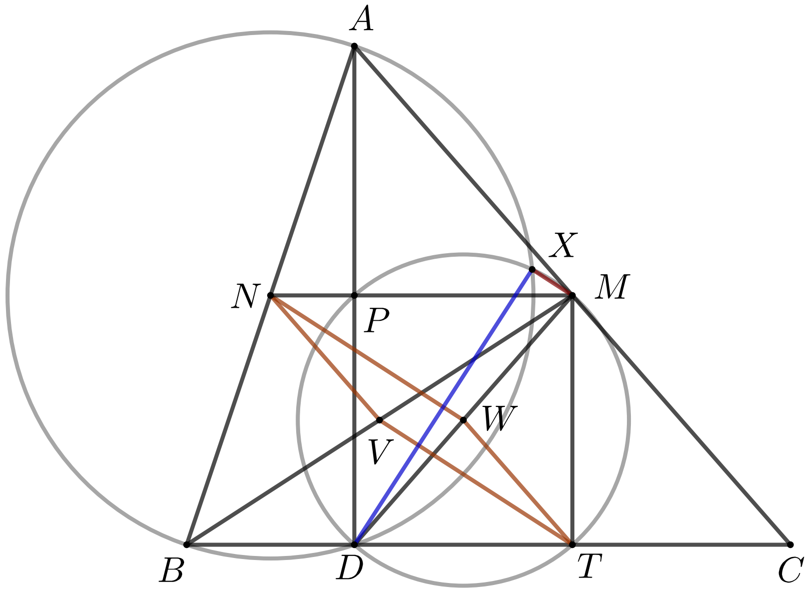
Soluţia 1: (*Mihai Miculiţa*)
Fie N şi P mijloacele segmentelor $[AB]$, respectiv $[AD]$. Atunci $P \in (NM)$, iar X este al doilea punct de intersecţie (diferit de D) dintre cercurile de diametre $[AB]$ şi $[DM]$. Cum $ABDX$ şi $DMXP$ sunt inscriptibile, avem $\sphericalangle XMN \equiv \sphericalangle XDP \equiv \sphericalangle XBA$, ceea ce arată că patrulaterul $BMXN$ este inscriptibil. Deducem că $\sphericalangle NMB \equiv \sphericalangle NXB \equiv \sphericalangle NBX$. Pe de altă parte, $NM \parallel BC$ implică $\sphericalangle MBC \equiv \sphericalangle NMB \equiv \sphericalangle NMX$, de unde concluzia.

Altă finalizare: Patrulaterul $BMXN$ este inscriptibil, iar coardelor congruente $[NB]$ şi $[NX]$ le corespund arce congruente, deci $\sphericalangle XMN \equiv \sphericalangle XMB \equiv \sphericalangle MBC$.



Remarcă: (*Rareş Spineanu*) Cele de mai sus arată că $(BX$ e simediană în triunghiul ABC.
Remarcă: (*Mihai Miculiţa*) Punctele X şi C se găseasc de o parte şi de alta a dreptei BM dacă şi numai dacă $AB < BC$.

Soluţia 2: Cercul de diametru $[MD]$ trece prin mijloacele P, respectiv T, ale segmentelor $[AD]$, respectiv $[CD]$. Dacă notăm cu N, V şi W mijloacele segmentelor $[AB]$, $[BM]$, respectiv $[DM]$, atunci $[NV]$ şi $[WT]$ sunt paralele (sau coliniare), fiind paralele cu AC şi egale (cu jumătate din AM, respectiv CM). Atunci $NWTV$ este paralelogram, deci $TV \parallel NW$. Dar $NW \perp XD$ (linia centrelor este perpendiculară pe axa radicală a cercurilor de diametre $[AB]$ şi $[MD]$), deci $TV \parallel NW \parallel XM$. În triunghiul dreptunghic BTM avem $VT = VB = VM$, deci $\sphericalangle XMN \equiv \sphericalangle VTB \equiv \sphericalangle VBT \equiv \sphericalangle NMB$, de unde concluzia.



Am primit soluţii de la: *Cezara Danciu, Radu Stoleriu, Andrei Pană, Emanuel Mazăre, Ana Duguleanu, Francesca Balaur, Radu Şerban* şi *Ştefan Gobej.*

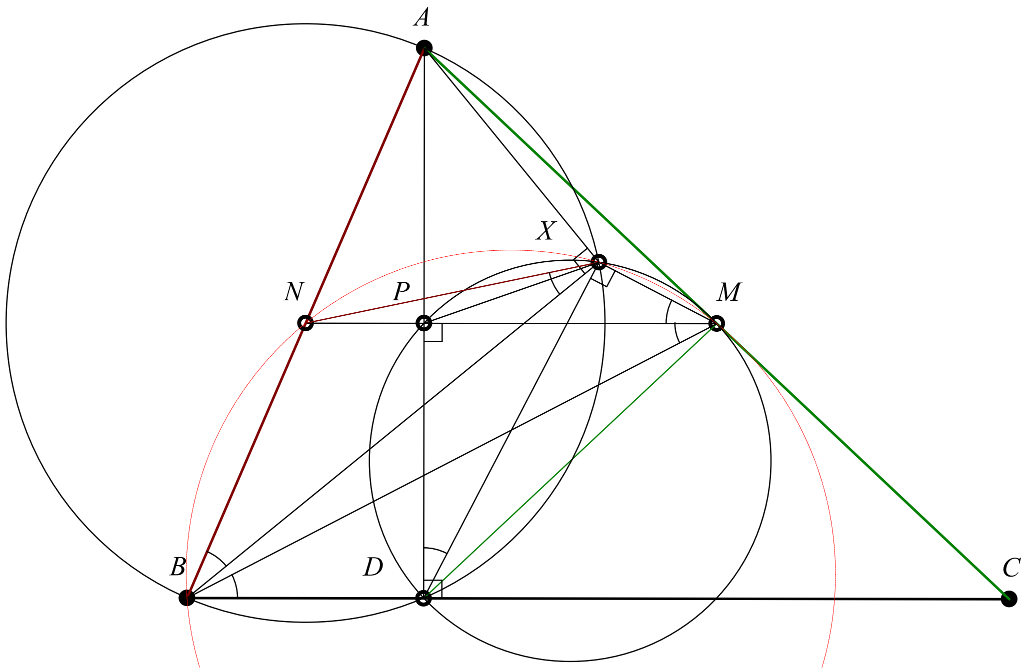
Problem of the week no. 241

Let ABC be an acute triangle in which D is the foot of the altitude dropped from A and M is the midpoint of $[AC]$. Assuming there is a point X such that $\widehat{AXB} = \widehat{DXM} = 90°$ and points X and C are on different sides of the line BM, prove that $\widehat{XMB} = 2 \cdot \widehat{MBC}$.

EGMO training, Italy, 2021

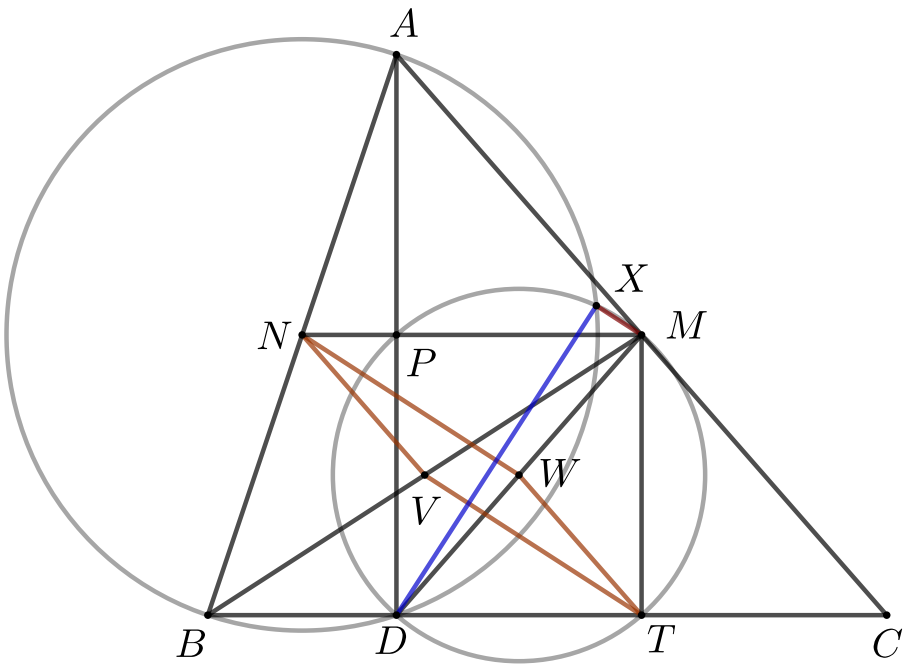
Solution 1: (*Mihai Miculiţa*)
FLet N and P be the midpoints of line segments $[AB]$ and $[AD]$, respectively. Then $P \in (NM)$, and X is the second point of intersection (the first one being D) between circles of diameters $[AB]$ and $[DM]$. As quadrilaterals $ABDX$ and $DMXP$ are cyclic, $\sphericalangle XMN \equiv \sphericalangle XDP \equiv \sphericalangle XBA$, which shows that quadrilateral $BMXN$ is also cyclic. It follows that $\sphericalangle NMB \equiv \sphericalangle NXB \equiv \sphericalangle NBX$. On the other hand, $NM \parallel BC$ means $\sphericalangle MBC \equiv \sphericalangle NMB \equiv \sphericalangle NMX$, which leads to the conclusion.

Another ending: Quadrilateral $BMXN$ is cyclic. To equal chords $[NB]$ and $[NX]$ correspond equal arcs, hence $\sphericalangle XMN \equiv \sphericalangle XMB \equiv \sphericalangle MBC$.



Remark: (*Rareş Spineanu*) The above shows that $(BX$ is the symmedian in triangle ABC.
Remark: (*Mihai Miculiţa*) Points X and C are on different sides of the line BM if and only if $AB < BC$.

Solution 2: The circle of diameter $[MD]$ passes through the midpoints P and T of line segments $[AD]$ and $[CD]$. If N, V and W are the midpoints of $[AB]$, $[BM]$,and $[DM]$, respectively, then $[NV]$ and $[WT]$ are parallel to AC and equal (to half of AM and CM, respectively). Thus $NWTV$ is a parallelogram, therefore $TV \parallel NW$. But $NW \perp XD$ (the radical axis of the circles of diameters $[AB]$ şi $[MD]$ is perpendicular to the center line of the two circles), hence $TV \parallel NW \parallel XM$. In the right triangle BTM we have $VT = VB = VM$, hence $\sphericalangle XMN \equiv \sphericalangle VTB \equiv \sphericalangle VBT \equiv \sphericalangle NMB$ and the conclusion.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