Problema săptămânii 238

Arătaţi că dacă $a, b, c > 0$ şi $abc \geq 1$, atunci

$$\frac{1}{1+b+c} + \frac{1}{1+c+a} + \frac{1}{1+a+b} \leq 1.$$

Soluţia 1: Eliminând numitorii se ajunge la $2(a+b+c) \leq ab(a+b) + bc(b+c) + ca(c+a)$, adică $(a+b+c)(ab+bc+ca-2) \geq 3$. Ori $a+b+c \geq 3\sqrt[3]{abc}$ şi $ab+bc+ca-2 \geq 3\sqrt[3]{a^2b^2c^2}-2 \geq 3-2 = 1$. Egalitatea are loc dacă $a = b = c = 1$.

Soluţia 2: Fie $x, y, z > 0$ astfel încât $a = x^3$, $b = y^3$, $c = z^3$. Folosim inegalitatea $x^3 + y^3 \geq xy(x+y)$ care revine la $(x-y)(x^2-y^2) \geq 0$, inegalitate adevărată, satisfăcută cu egalitate dacă $x = y$. Atunci

$$\frac{1}{1+x^3+y^3} + \frac{1}{1+y^3+z^3} + \frac{1}{1+z^3+x^3} \leq$$

$$\frac{1}{1+xy(x+y)} + \frac{1}{1+yz(y+z)} + \frac{1}{1+zx(z+x)} \leq$$

$$\frac{1}{1+\dfrac{x+y}{z}} + \frac{1}{1+\dfrac{y+z}{x}} + \frac{1}{1+\dfrac{x+z}{y}} =$$

$$\frac{z}{x+y+z} + \frac{x}{x+y+z} + \frac{y}{x+y+z} = 1.$$

În inegalitatea din enunţ avem egalitate dacă $x = y = z$ şi $xyz = 1$, deci pentru $a = b = c = 1$.

Soluţia 3: (*Ana Duguleanu*)

$$\frac{1}{1+b+c} + \frac{1}{1+c+a} + \frac{1}{1+a+b} = \frac{a+2}{(a+1+1)(1+b+c)} + \frac{b+2}{(b+1+1)(1+c+a)}$$

$$+ \frac{c+2}{(c+1+1)(1+a+b)} \overset{CBS}{\leq} \frac{a+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} + \frac{b+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} +$$

$\dfrac{c+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} = \dfrac{a+b+c+6}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} \leq 1$. Ultima inegalitate revine la $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 3$ şi rezultă din inegalitatea mediilor şi condiţia $abc \geq 1$.

Am primit soluţii de la: *David Ghibu, Francesca Balaur, Andrei Pană, Cezara Danciu, Ştefan Gobej, Marian Cucoaneş, Corneliu Mănescu-Avram, Emanuel Mazăre, Ana Duguleanu, Nicuşor Zlota* şi *Adrian Zanca.*

Problem of the week no. 238

Positive real numbers a, b, c satisfy $abc \geq 1$. Prove that

$$\frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b} \leq 1.$$

Solution 1: After some computations, the inequality reduces to $2(a+b+c) \leq ab(a+b)+bc(b+c)+ca(c+a)$, i.e. $(a+b+c)(ab+bc+ca-2) \geq 3$. But $a+b+c \geq 3\sqrt[3]{abc}$ and $ab+bc+ca-2 \geq 3\sqrt[3]{a^2b^2c^2}-2 \geq 3-2=1$. Equality holds if $a=b=c=1$.

Solution 2: Let $x, y, z > 0$ such that $a=x^3$, $b=y^3$, $c=z^3$. We use the inequality $x^3+y^3 \geq xy(x+y)$, which can be written as $(x-y)(x^2-y^2) \geq 0$. This is true because the two factors have the same sign. Equality holds if $x=y$. Then

$$\frac{1}{1+x^3+y^3}+\frac{1}{1+y^3+z^3}+\frac{1}{1+z^3+x^3} \leq$$

$$\frac{1}{1+xy(x+y)}+\frac{1}{1+yz(y+z)}+\frac{1}{1+zx(z+x)} \leq$$

$$\frac{1}{1+\dfrac{x+y}{z}}+\frac{1}{1+\dfrac{y+z}{x}}+\frac{1}{1+\dfrac{x+z}{y}} =$$

$$\frac{z}{x+y+z}+\frac{x}{x+y+z}+\frac{y}{x+y+z}=1.$$

In the given inequality, equality holds if $x=y=z$ and $xyz=1$, i.e. for $a=b=c=1$.

Solution 3: (*Ana Duguleanu*)

$$\frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b}=\frac{a+2}{(a+1+1)(1+b+c)}+\frac{b+2}{(b+1+1)(1+c+a)}$$
$$+\frac{c+2}{(c+1+1)(1+a+b)} \overset{CBS}{\leq} \frac{a+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}+\frac{b+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}+$$
$$\frac{c+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}=\frac{a+b+c+6}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} \leq 1.$$ The last inequality comes down to $\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3$ and follows from the AM-GM inequality and from the condition $abc \geq 1$.