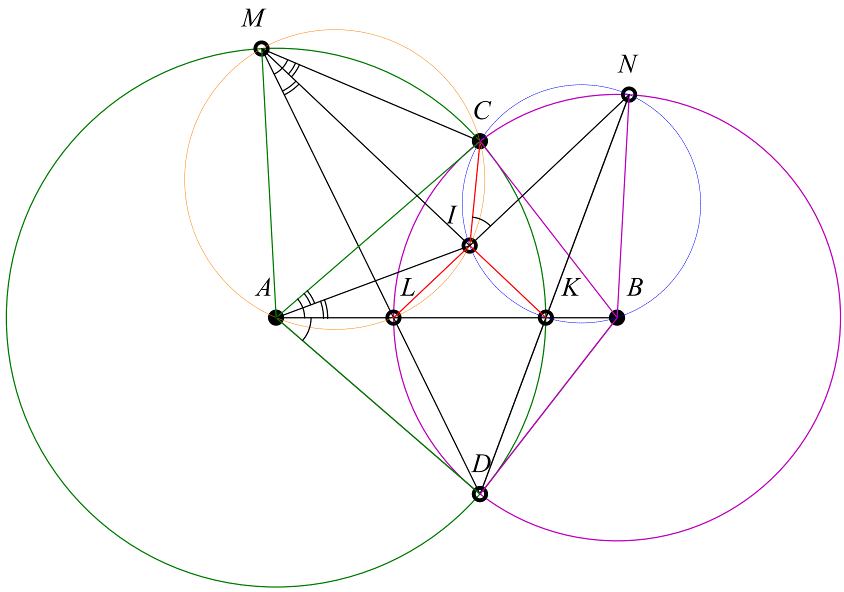
Problema săptămânii 233

Două cercuri, (A, r_a) şi (B, r_b), având centrele A, respectiv B, se intersectează în punctele C şi D. Notăm cu K şi L intersecţiile segmentului $[AB]$ cu cercurile (A, r_a), respectiv (B, r_b), cu M al doilea punct de intersecţie a dreptei DL cu cercul (A, r_a) şi cu N al doilea punct de intersecţie a dreptei DK cu cercul (B, r_b). Dacă $\{I\} = KM \cap LN$, arătaţi că I este centrul cercului înscris în triunghiul ABC.

Constantin Knop, Turneul de toamnă al oraşelor din Rusia, 2020



SOLUŢIE (*Mihai Miculiţa*): Întrucât cercurile $(A;r_a) = \odot DKM$ şi $(B;r_b) = \odot DNL$, cel de al doilea punct comun al lor, punctul C – este punctul lui MIQUEL al patrulatrului complet $DKILMN$. Aşa că: $C \in \odot LIM, \odot KNI \Rightarrow$ patrulaterele $CILM$ şi $CIKN$ – sunt inscriptibile, iar din:

$$(A;r_a) \cap (B;r_b) = \{C;D\} \Rightarrow \left.\begin{cases}[AC] \equiv [AD] \\ [BC] \equiv [BC]\end{cases}\right\} \Rightarrow ADBC - deltoid \Rightarrow \widehat{BAC} \equiv \widehat{BAD} \Rightarrow$$

$$\Rightarrow m\left(\widehat{BAC}\right) = m\left(\widehat{BAD}\right) = \frac{1}{2} \cdot m\left(\widehat{CAD}\right) \Rightarrow m\left(\overset{\frown}{KC}\right) = m\left(\overset{\frown}{KD}\right) = \frac{1}{2} \cdot m\left(\overset{\frown}{CD}\right) \text{ (1)} \Rightarrow$$

$$\left.\left.\begin{array}{r} \Rightarrow \widehat{BAC} \equiv \widehat{BAD} \equiv \underline{\widehat{CMD}} \\ CILM - inscriptibil \Rightarrow \widehat{CIN} \equiv \underline{\widehat{CMD}} \end{array}\right\} \Rightarrow \widehat{LAC}\left(= \widehat{BAC}\right) \equiv \widehat{CIN} \Rightarrow ALIC - inscriptibil \atop CILM - inscriptibil\right\} \Rightarrow$$

$$\Rightarrow \odot ALIC = \odot CILM \Rightarrow ALICM - pentagon\ insciptibil. \quad (2)$$

Pe de altă parte, din:

$$\left.\left.\begin{array}{r} KCMD - inscriptibil \\ \overset{\frown}{KC} \equiv \overset{\frown}{KD} \text{ (1)} \end{array}\right\} \Rightarrow \widehat{IMC}\left(= \widehat{KMC}\right) \equiv \widehat{IML}\left(= \widehat{KMD}\right) \Rightarrow \overset{\frown}{IC} \equiv \overset{\frown}{IL} \atop ALIC - inscriptibil \text{ (2)}\right\} \Rightarrow \widehat{IAC} \equiv \widehat{IAB}\left(= \widehat{IAL}\right) \Rightarrow$$

$$\Rightarrow \boxed{(AI - este\ bisectoarea\ \widehat{CAB}} \text{ şi în mod analog arătăm că } \boxed{(BI - este\ bisectoarea\ \widehat{CAB}}. \blacksquare$$

Soluţia, în engleză, de mai jos, este diferită.

Am primit soluţii de la: *Francesca Balaur, David Ghibu, Cezara Danciu, Emanuel Mazăre, Radu Şerban, Ştefan Gobej, Ana Duguleanu* şi *Tashi Diaconescu.*

Problem of the week no. 233

Two circles, (A, r_a) and (B, r_b), centered at A and B, respectively, intersect at C and D. Let K and L be the intersections of the line segment $[AB]$ with circles (A, r_a) and (B, r_b), respectively. The line DL meets the circle (A, r_a) again at M; line DK meets the circle (B, r_b) again at N. If $\{I\} = KM \cap LN$, prove that I is the incenter of triangle ABC.

Constantin Knop, Tournament of Towns, 2020

Solution: As $\sphericalangle CAK = \frac{1}{2}\sphericalangle CAD = \sphericalangle DMC$, i.e. $\sphericalangle LAC = \sphericalangle LMC$, quadrilateral $LAMC$ is cyclic. Similarly, $KBNC$ is cyclic. It easily follows that isosceles triangles CAM and LBC are similar as are triangles CAK and CBN. Thus, $\sphericalangle MAK = \sphericalangle NBL$, and $\sphericalangle MCK = 180° - \frac{1}{2}\sphericalangle MAK = 180° - \frac{1}{2}\sphericalangle NBL = \sphericalangle LCN \quad (*)$. Also, from the similarities above, we get $\dfrac{CL}{CB} = \dfrac{CM}{CA}$ and $\dfrac{CN}{CB} = \dfrac{CK}{CA}$. It follows that $\dfrac{CL}{CN} = \dfrac{CM}{CK}$. Together with $(*)$, this means that triangles LCN and MCK are similar, therefore $\sphericalangle CMI = \sphericalangle CMI$ and $\sphericalangle CNI = \sphericalangle CKI$. This means that quadrilaterals $CMLI$ and $CNKI$ are cyclic. We conclude that points C, M, A, L, I are co-cyclic and so are C, N, B, K, I. Then $\sphericalangle LAI = \sphericalangle LMI = \sphericalangle DMK = \sphericalangle KMC = \sphericalangle IMC = \sphericalangle IAC$, which shows that $(AI$ is the bisector of angle $\sphericalangle BAC$. Similarly, $(BI$ is the bisector of angle $\sphericalangle ABC$, which makes I the incenter of triangle ABC.