Problema săptămânii 234

Fie n un număr natural nenul. Determinaţi toate n-uplurile de numere reale pozitive $(x_1, x_2, \ldots, x_n)$ cu proprietatea

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\ldots\left(x_n + \frac{1}{x_1}\right) = \left(x_1^2 + \frac{1}{x_2^2}\right)\left(x_2^2 + \frac{1}{x_3^2}\right)\ldots\left(x_n^2 + \frac{1}{x_1^2}\right).$$

antrenament Franţa, 2020

Soluţie:

Notăm $x_{n+1} = x_1$. Din inegalitatea dintre media pătratică şi cea aritmetică avem, pentru orice $k \in \{1, 2, \ldots, n\}$,

$$x_k^2 + \frac{1}{x_{k+1}^2} \geq \frac{1}{2}\left(x_k + \frac{1}{x_{k+1}}\right)^2.$$

Prin înmulţirea acestor inegalităţi obţinem că

$$\prod_{k=1}^{n}\left(x_k + \frac{1}{x_{k+1}}\right) = \prod_{k=1}^{n}\left(x_k^2 + \frac{1}{x_{k+1}^2}\right) \geq \frac{1}{2^n}\prod_{k=1}^{n}\left(x_k + \frac{1}{x_{k+1}}\right)^2,$$

de unde

$$\prod_{k=1}^{n}\left(x_k + \frac{1}{x_{k+1}}\right) \leq 2^n. \qquad (1)$$

Pe de altă parte, din inegalitatea mediilor, $x_k + \dfrac{1}{x_{k+1}} \geq 2\sqrt{\dfrac{x_k}{x_{k+1}}}, \forall\, k \in \{1, 2, \ldots, n\}$.

Înmulţind aceste n inegalităţi obţinem că

$$\prod_{k=1}^{n}\left(x_k + \frac{1}{x_{k+1}}\right) \geq 2^n. \qquad (2)$$

Din (1) şi (2) deducem că $(x_1, x_2, \ldots, x_n)$ trebuie să satisfacă inegalitatea (2) cu egalitate, ceea ce se întâmplă dacă şi numai dacă $x_k = \dfrac{1}{x_{k+1}}, \forall\, k \in \{1, 2, \ldots, n\}$.

Dacă n este impar, obţinem că $x_1 = x_2 = \ldots = x_n = 1$ este unica soluţie.

Dacă n este par, soluţiile sunt n-uplurile de forma $\left(a, \frac{1}{a}, a, \frac{1}{a}, \ldots, \frac{1}{a}, a, \frac{1}{a}\right)$, cu $a > 0$ arbitrar.

Am primit soluţii de la: *David Ghibu, Cezara Danciu, Emanuel Mazăre, Francesca Balaur, Carol Luca Gasan, Ana Duguleanu, Ştefan Gobej* şi *Ana Boiangiu.*

Problem of the week no. 234

Let n be a positive integer. Determine all n-tuples of positive real numbers that satisfy

$$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\cdots\left(x_n+\frac{1}{x_1}\right)=\left(x_1^2+\frac{1}{x_2^2}\right)\left(x_2^2+\frac{1}{x_3^2}\right)\cdots\left(x_n^2+\frac{1}{x_1^2}\right).$$

Training problem, France, 2020

Solution:

Put $x_{n+1}=x_1$. From the inequality between the arithmetic and quadratic means, we have, for all $k\in\{1,2,\ldots,n\}$,

$$x_k^2+\frac{1}{x_{k+1}^2}\geq\frac{1}{2}\left(x_k+\frac{1}{x_{k+1}}\right)^2.$$

Multiplying together these inequalities yields

$$\prod_{k=1}^{n}\left(x_k+\frac{1}{x_{k+1}}\right)=\prod_{k=1}^{n}\left(x_k^2+\frac{1}{x_{k+1}^2}\right)\geq\frac{1}{2^n}\prod_{k=1}^{n}\left(x_k+\frac{1}{x_{k+1}}\right)^2,$$

hence

$$\prod_{k=1}^{n}\left(x_k+\frac{1}{x_{k+1}}\right)\leq 2^n. \qquad (1)$$

On the other hand, from the AM-GM inequality, $x_k+\dfrac{1}{x_{k+1}}\geq 2\sqrt{\dfrac{x_k}{x_{k+1}}}$, $\forall\, k\in\{1,2,\ldots,n\}$. Multiplying together these inequalities yields

$$\prod_{k=1}^{n}\left(x_k+\frac{1}{x_{k+1}}\right)\geq 2^n. \qquad (2)$$

From (1) and (2) it follows that $(x_1,x_2,\ldots,x_n)$ must satisfy inequality (2) with equality, which happens if and only if $x_k=\dfrac{1}{x_{k+1}}$, $\forall\, k\in\{1,2,\ldots,n\}$.

If n is odd, the only solution is $x_1=x_2=\ldots=x_n=1$.

If n is even, the sought solutions are $\left(a,\frac{1}{a},a,\frac{1}{a},\ldots,\frac{1}{a},a,\frac{1}{a}\right)$, with any $a>0$.