Problema săptămânii 222

Arătaţi că dacă a, b, c, x, y, z sunt numere reale pozitive şi $x + y + z = 1$, atunci

$$ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)} \leq a + b + c.$$

Soluţia 1:
Din inegalitatea Cauchy-Buniakowsky-Schwarz,

$$ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)}$$

$$\leq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{x^2 + y^2 + z^2} + \sqrt{2(ab + bc + ca)} \cdot \sqrt{2(xy + yz + zx)}$$

$$\leq \sqrt{x^2 + y^2 + z^2 + 2(xy + yz + zx)} \cdot \sqrt{a^2 + b^2 + c^2 + 2(ab + bc + ca)}$$

$$= (x + y + z)(a + b + c) = a + b + c.$$

Egalitatea are loc atunci când $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ şi $\frac{x^2 + y^2 + z^2}{2(xy + yz + zx)} = \frac{a^2 + b^2 + c^2}{2(ab + bc + ca)}$, ceea ce revine la $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

Soluţia 2: (*Ana Duguleanu, Tashi Diaconescu*)
Din inegalitatea Cauchy-Buniakowsky-Schwarz avem

$$(a + b + c)^2 = (a + b + c)^2(x + y + z)^2 =$$

$$[a^2 + b^2 + c^2 + 2(ab + bc + ca)] \cdot [x^2 + y^2 + z^2 + 2(xy + yz + zx)] \overset{CBS}{\geq}$$

$$(ax + by + cz + \sqrt{(2ab + 2bc + 2ca) \cdot (2xy + 2yz + 2zx)})^2,$$

de unde rezultă inegalitatea din enunţ.
Egalitatea are loc atunci când $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{\sqrt{2(ab + bc + ca)}}{\sqrt{2xy + 2yz + 2zx}}$.

Dacă $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = k$, atunci şi $\frac{\sqrt{2(ab + bc + ca)}}{\sqrt{2xy + 2yz + 2zx}} = k$, deci egalitatea are loc dacă şi numai dacă $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

Am primit soluţii de la *Carol Luca Gasan, Cezara Danciu, Francesca Balaur, Ştefan Gobej, Radu Stoleriu, Ana Duguleanu, Emanuel Mazăre, Andrei Pană, Tashi Diaconescu* şi *Ana Boiangiu.*

Problem of the week no. 222

Let a, b, c, x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)} \leq a + b + c.$$

Solution 1:
From the Cauchy-Buniakowsky-Schwarz inequality,

$$ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)}$$

$$\leq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{x^2 + y^2 + z^2} + \sqrt{2(ab + bc + ca)} \cdot \sqrt{2(xy + yz + zx)}$$
$$\leq \sqrt{x^2 + y^2 + z^2 + 2(xy + yz + zx)} \cdot \sqrt{a^2 + b^2 + c^2 + 2(ab + bc + ca)}$$
$$= (x + y + z)(a + b + c) = a + b + c.$$

Equality holds if $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ and $\frac{x^2 + y^2 + z^2}{2(xy + yz + zx)} = \frac{a^2 + b^2 + c^2}{2(ab + bc + ca)}$, which comes down to $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

Solution 2: (*Ana Duguleanu, Tashi Diaconescu*)
From the Cauchy-Buniakowsky-Schwarz inequality it follows that

$$(a + b + c)^2 = (a + b + c)^2(x + y + z)^2 =$$

$$[a^2 + b^2 + c^2 + 2(ab + bc + ca)] \cdot [x^2 + y^2 + z^2 + 2(xy + yz + zx)] \overset{CBS}{\geq}$$
$$(ax + by + cz + \sqrt{(2ab + 2bc + 2ca) \cdot (2xy + 2yz + 2zx)})^2,$$

and the requested inequality follows immediately.
Equality holds if and only if $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{\sqrt{2(ab + bc + ca)}}{\sqrt{2xy + 2yz + 2zx}}$.

If $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = k$, then $\frac{\sqrt{2(ab + bc + ca)}}{\sqrt{2xy + 2yz + 2zx}} = k$, therefore equality holds if and only if $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.