Degenerate Circles

NATHAN RAMESH[*]

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Power of a Point

Let P be a point and ω be a circle with center O and radius r. We define the **power** of the point P with respect to the circle ω to be $OP^2 - r^2$, and we denote this by $\text{pow}(P, \omega)$.

Theorem (Power of a Point). Suppose a line ℓ through P intersects a circle ω at two points A and B. Then $PA \cdot PB = \text{pow}(P, \omega)$.

In fact, the converse also holds.

Theorem (Converse of Power of a Point). Suppose P, A, B, C, D are points with P the intersection of AB and CD. Assume that P is on either both or neither of the segments AB and CD. Then $PA \cdot PB = PC \cdot PD$ iff A, B, C, D are concyclic.

Problems

Problem 1. DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the center of the circle. Join OE and extend OE to cut the circle at C. Given $EC = 1$, find the radius of the circle.

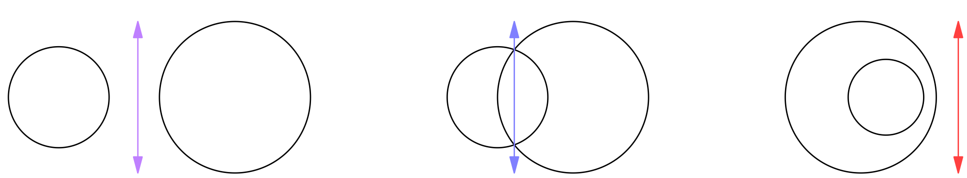
Problem 2. For each parabola $y = x^2 + px + q$, which intersects the coordinate axes in three different points, consider the circle passing through these three points. Prove that all these circles have a common point.

Problem 3. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Radical Axis

We define the **radical axis** of two circles ω_1 and ω_2 to be the locus of all points P such that $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2)$.

Theorem (Radical Axis). The radical axis of two circles ω_1 and ω_2 with centers O_1 and O_2 respectively is a line perpendicular to AB.



[*]Special thanks to Jason Chen for help with diagrams.

Sketch. Use Cartesian Coordinates.

Remark. Observe that using Cartesian Coordinates, none of the algebra breaks if one or more of the radii are 0. This will be important later.

Theorem (Radical Axis Concurrence). The pairwise radical axes of three circles are either concurrent or pairwise parallel. The concurrence point is referred to as the **radical center** of the three circles.

Proof. Call the circles ω_1, ω_2, and ω_3. Suppose two of the radical axes intersect at a point P. WLOG say P is the intersection of the radical axis of ω_1 and ω_2 and the radical axis of ω_2 and ω_3. By the definition of radical axis, $\text{pow}(P, \omega_1) = \text{pow}(P, \omega_2) = \text{pow}(P, \omega_3)$, hence P lies on the radical axis of ω_1 and ω_3. In this case such a radical center exists and the former criteria holds. Suppose instead that such a point P did not exist. It follows that the radical axes must be pairwise parallel and thus the latter criteria holds.

We will look at a quick configuration before moving on to problems.

Configuration. Let ω_1 and ω_2 be two circles that intersect at points A and B. Suppose CD is an external tangent to both circles, tangent to ω_1 at C and ω_2 at D. Prove that AB bisects CD.

Solution. Note that $\text{pow}(A, \omega_1) = \text{pow}(A, \omega_2) = 0$ and $\text{pow}(B, \omega_1) = \text{pow}(B, \omega_2) = 0$. It follows that A and B both lie on that radical axis of ω_1 and ω_2, and hence AB is the radical axis. Letting P be the intersection point, we have $PC^2 = \text{pow}(P, \omega_1) = \text{pow}(P, \omega_2) = PD^2$, and the result follows.

Problems

Problem 1 (JMO 2012). Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

Problem 2 (IMO 1995). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Problem 3 (USAMO 2009). Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

Degenerate Circles

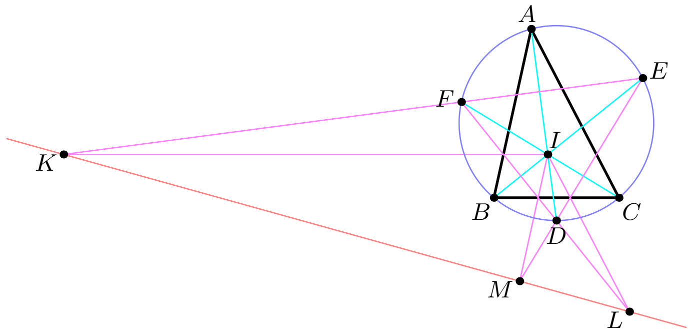
We will look at a special case of the radical axis theorem. In particular, we will examine the case where the radius of one or more of the circles is *zero*.

Problem (Circumcenter). In a triangle $\triangle ABC$, prove that the perpendicular bisectors of AB, BC, and CA concur.

Solution. Let ω_A be the circle of radius zero centered at A and define ω_B and ω_C similarly. Note that the perpendicular bisector of AB is the radical axis of ω_A and ω_B. Similar results hold for the other two perpendicular bisectors. Applying the radical axis concurrence theorem to ω_A, ω_B, and ω_C yields the result.

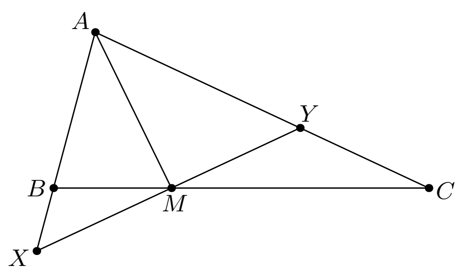
Problem (Balkan MO 2015). Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB

intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear.

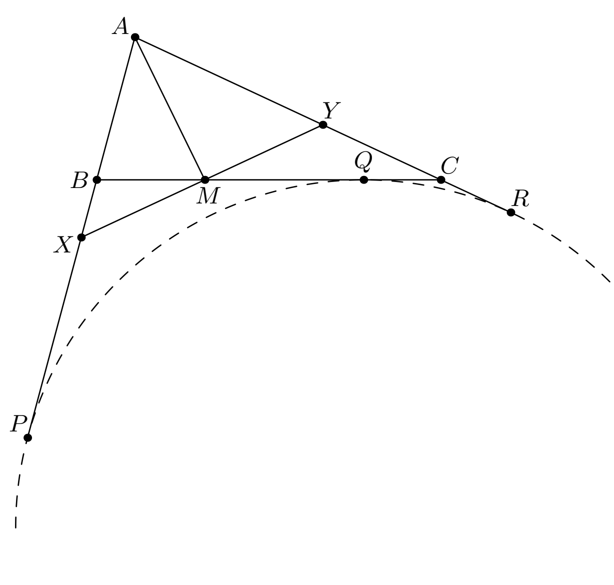


Solution. Note that $\angle KEI = \angle FEI = \angle FEB = \angle FCB = \angle FIK$. It follows from AA similarity that $\triangle KFI \sim \triangle KIE$, hence $KF \cdot KE = KI^2$. Now we have that K lies on the radical axis of ω and the circle centered at I with radius zero. Similarly, L and M also lie on this line, hence the colinearity.

Problem (ARMO 2011). The perimeter of triangle $\triangle ABC$ is 4. Point X is marked on ray AB and point Y is marked on ray AC such that $AX = AY = 1$. Suppose segment BC intersects segment XY at point M. Prove that perimeter of one of triangles $\triangle ABM$ or $\triangle ACM$ is 2.



Solution. Consider the A-excircle of $\triangle ABC$, and call it ω. Let P, Q, R be the tangency points of ω with lines AB, BC, CA, respectively. Note that $AP = AR = 2$, so it follows that XY is the radical axis of ω and the circle of radius zero centered at A. Since M lies on this radical axis, it follows that $MA = MQ$. WLOG suppose Q lines on segment MC. Then the perimeter of $\triangle ABM = AB + BM + MA = AB + BM + MQ = AB + BQ = AB + BP = 2$, as desired.



Problems

Enjoy the following problems, arranged roughly by difficulty.*

Problem 1. Let ABC be an acute angled triangle. Let ω be its incircle and I be its center. Let the contact points of ω with BC,CA and AB be D,E,F respectively. Let $IB \cap DF = P$ and M be the mid-point of BP. From M, draw a line ℓ parallel to DF and let K be any point on ℓ. From K draw tangent KS to ω with S on ω. Then prove that K is the circumcenter of $\triangle BPS$.

Problem 2. Let I and O be the incenter and circumcenter of a scalene triangle ABC. Let the line passing through I perpendicular to AI intersect BC at A'. Points B' and C' are defined similarly. Prove that A', B', C' lie on a line perpendicular to IO.

Problem 3 (College Geometry). Let A be a point outside a circle ω and let P and P' be the points of tangency to ω of the tangents from A and let R and R' be the midpoints of AP and AP' respectively. Choose a point T on the segment RR' and draw its tangents to ω with points of tangency F and F'. The line FF' cuts RR' in U. Prove that $\triangle TAU$ is a right triangle.

Problem 4 (Iran TST 2011). In acute triangle ABC angle B is greater than C. Let M is midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are midpoint of ME and MD respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

Problem 5 (ISL 2009). Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Problem 6 (Geolmypiad Summer 2015). Let ω_1, ω_2 be non-intersecting, congruent circles with centers O_1, O_2 and let P be in the exterior of both of them. The tangents from P to ω_1 meet ω_1 at A_1, B_1 and define A_2, B_2 similarly. If lines A_1B_1, A_2B_2 meet at Q show that the midpoint of PQ is equidistant from O_1, O_2.

Problem 7. (CGMO 2015) Let Γ_1 and Γ_2 be two non-overlapping circles. A, C are on Γ_1 and B, D are on Γ_2 such that AB is an external common tangent to the two circles, and CD is an internal common tangent to the two circles. AC and BD meet at E. F is a point on Γ_1, the tangent line to Γ_1 at F meets the perpendicular bisector of EF at M. MG is a line tangent to Γ_2 at G. Prove that $MF = MG$.

Problem 7. Let $\triangle ABC$ be an acute triangle with $AB \neq AC$ and orthocenter H. In addition, let M, N be the midpoints of AB and AC, respectively. Say D is the projection of A onto BC and (E) is the circumcircle of the triangle $\triangle DMN$. Suppose HM and HN intersect (E) again at P and Q, respectively. Furthermore, suppose DP and DQ intersect BH, CH at K and L, respectively. Prove that $EH \perp KL$.

*If you want hints, message me on Facebook.