ASUPRA UNEI IDENTITĂŢI

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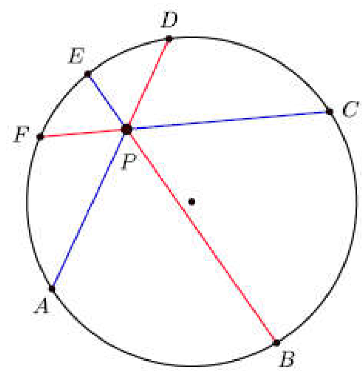
Fie $T-$punctul lui TORICELLI al unui triunghi oarecare $ABC,$ adică punctul având proprietatea, că: $\measuredangle BTC=\measuredangle CTA=\measuredangle ATB=120^0.$ Ne propunem să calculăm suma: $|TA|+|TB|+|TC|$ în funcţie de elementele triunghiului $ABC.$



Notând cu $x:=|TA|,\ y:=|TB|$ şi cu $z:=|TC|,$ avem:

$$\left.\begin{aligned}a^2&=y^2+z^2-2yz\cos 120^0=y^2+z^2+yz\\ b^2&=x^2+z^2+xz\\ c^2&=x^2+y^2+xy\end{aligned}\right\}\Rightarrow a^2+b^2+c^2=2\left(x^2+y^2+z^2\right)+xy+yz+zx=$$

$$=2(x+y+z)^2-3(xy+yz+zx)=2(x+y+z)^2-3\cdot\left(S_{ATB}+S_{BTC}+S_{ATC}\right)\cdot\frac{4}{\sqrt{3}}=$$

$$=2(x+y+z)^2-4\sqrt{3}\cdot S_{ATB}\Rightarrow(x+y+z)^2=\frac{a^2+b^2+c^2+4\sqrt{3}\cdot S}{2}=$$

$$=\frac{4R^2\cdot\left(\sin^2 A+\sin^2 B+\sin^2 C\right)+2\sqrt{3}R^2\cdot\left(\sin 2A+\sin 2B+\sin 2C\right)}{2}=$$

$$=\frac{4R^2\cdot\left(\frac{1-\cos 2A}{2}+\frac{1-\cos 2B}{2}+\frac{1-\cos 2C}{2}\right)+2\sqrt{3}R^2\cdot\left(\sin 2A+\sin 2B+\sin 2C\right)}{2}=$$

$$=R^2\left[\sqrt{3}\left(\sin 2A+\sin 2B+\sin 2C\right)-\cos 2A-\cos 2B-\cos 2C+3\right]=$$

$$=R^2\left[ctg30^0\left(\sin 2A+\sin 2B+\sin 2C\right)-\cos 2A-\cos 2B-\cos 2C+3\right]=$$

$$=R^2\left[\frac{\sum\limits_{ciclic}\left(\sin 2A\cos 30^0-\sin 30^0\cos 2A\right)}{\sin 30^0}+3\right]=R^2\left[2\sum_{ciclic}\sin\left(2A-30^0\right)+3\right]\Rightarrow$$

$$\Rightarrow(TA+TB+TC)^2=R^2\left[2\sum_{ciclic}\sin\left(2A-30^0\right)+3\right].\quad(*)$$

Demonstrăm identitatea: $|PA|+|PC|+|PE|=|PB|+|PD|+|PF|$, notând cu:
$x:=\overset{\frown}{AB}, y:=\overset{\frown}{BC}, z:=\overset{\frown}{CD}, t:=\overset{\frown}{DE}, u:=\overset{\frown}{EF}$ şi cu: $v:=\overset{\frown}{FA} \Rightarrow x+t=y+u=z+v=120^0$
şi ţinând acum seama de relaţia (*), este sufficient să observăm că:

$$\sin\left(2\measuredangle AEC-30\right)+\sin\left(2\measuredangle CAE-30\right)+\sin\left(2\measuredangle ACE-30\right)=$$
$$=\sin\left(2\measuredangle FBD-30\right)+\sin\left(2\measuredangle BDF-30\right)+\sin\left(2\measuredangle DFB-30\right),$$

sau

$$\sin\left(x+y-30\right)+\sin\left(z+t-30\right)+\sin\left(u+w-30\right)=\sin\left(t+u-30\right)+\sin\left(x+w-30\right)+\sin\left(y+z-30\right)$$

identiate ce rezultă imediat, deoarece unghiurile sunt suplementare două cate două:

$$\left(x+y-30\right)+\left(t+u-30\right)=\left(x+t\right)+\left(y+y\right)-60=240-60=180$$
$$\left(z+t-30\right)+\left(x+w-30\right)=\left(x+t\right)+\left(z+w\right)-60=240-60=180$$
$$\left(u+w-30\right)+\left(y+z-30\right)=\left(y+u\right)+\left(z+w\right)-60=240-60=180$$

Cu aceasta, demonstraţia este incheiată.