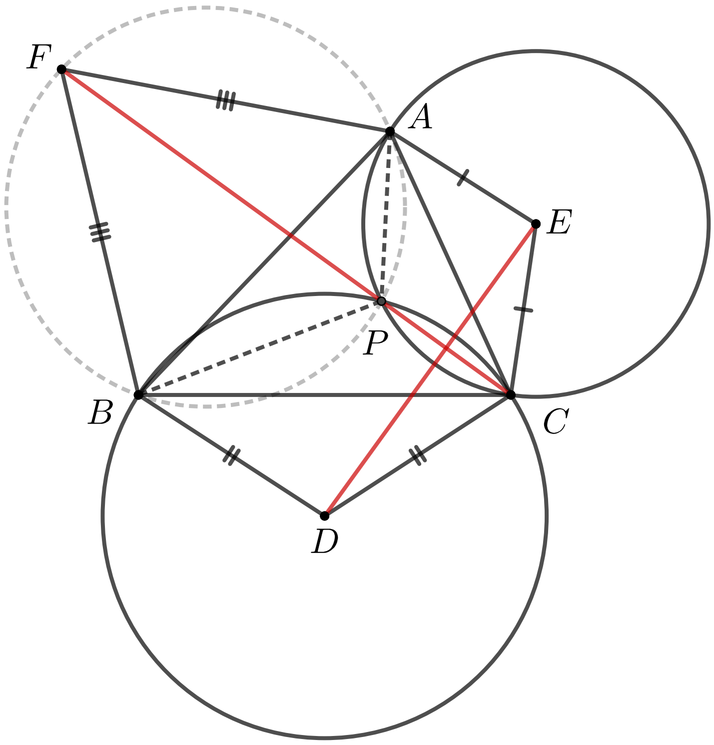
Problema săptămânii 205
Pe laturile triunghiului ABC se construiesc, în exterior, triunghiurile BCD, CAE şi ABF astfel încât $m(\sphericalangle BCD) = m(\sphericalangle CBD) = m(\sphericalangle EAC) = m(\sphericalangle ECA) = x$ şi $m(\sphericalangle FAB) = m(\sphericalangle FBA) = 90^\circ - x$. Demonstraţi că $DE \perp CF$.

Stanley Rabinowitz

Soluţie: Fie P al doilea punct de intersecţie a cercului de centru D şi rază DC cu cercul de centru E şi rază EC. (Dacă $C \in (DE)$, atunci $P = C$.) Deoarece $m(\sphericalangle APC) = 180^\circ - \dfrac{1}{2} m(\sphericalangle AEC) = 90^\circ + x$ şi, analog, $m(\sphericalangle BPC) = 90^\circ + x$, avem $m(\sphericalangle APB) = 180^\circ - 2x = 180^\circ - m(\sphericalangle AFB)$, deci patrulaterul $APBF$ este inscriptibil. Rezultă că $m(\sphericalangle APF) = m(\sphericalangle ABF) = 90^\circ - x$, deci $\sphericalangle APC$ şi $\sphericalangle APF$ sunt suplementare. Rezultă că punctele C, P şi F sunt coliniare. Asta înseamnă că FC este axa radicală a celor două cercuri amintite la început, deci este perpendiculară pe dreapta DE determinată de centrele acestora.



O soluţie pe o idee asemănătoare am primit de la *Radu Şerban*.

O altă idee (*Marin Hristov, Andrei Pană, Ana Duguleanu, David Ghibu*): concluzia este echivalentă cu relaţia $CD^2 + EF^2 = CE^2 + DF^2$. Calculând lungimile celor patru segmente implicate (cu teorema cosinusului), folosind că $\cos(90^\circ + t) = -\sin t$, relaţia de mai sus se reduce la teorema sinusurilor în triunghiul ABC.

Remarcă: (*Tashi Diaconescu*)
Afirmaţia rămâne adevărată şi dacă triunghiurile se construiesc către interior.

Tashi Diaconescu ne-a trimis o rezolvare scurtă folosind numere complexe.

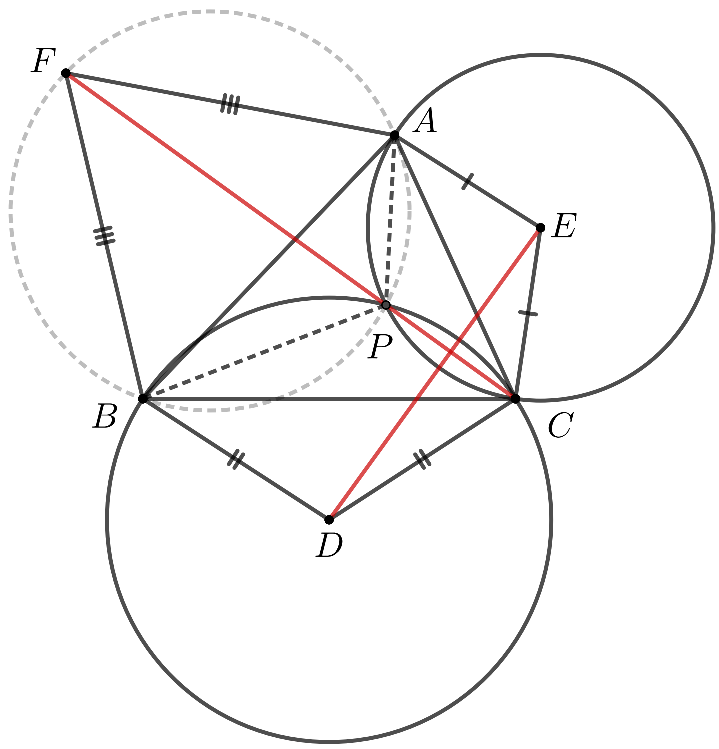
Problem of the week no. 205

On the sides of triangle ABC construct, externally, triangles BCD, CAE and ABF such that $\widehat{BCD} = \widehat{CBD} = \widehat{EAC} = \widehat{ECA} = x$ and $\widehat{FAB} = \widehat{FBA} = 90^\circ - x$. Prove that $DE \perp CF$.

Stanley Rabinowitz

Solution:

Let P be the second point of intersection of the circles centered at D and E that pass through C. (If $C \in (DE)$, then $P = C$.) Since $\sphericalangle APC = 180^\circ - \frac{1}{2}\sphericalangle AEC = 90^\circ + x$ and, similarly, $\sphericalangle BPC = 90^\circ + x$, we have $\sphericalangle APB = 180^\circ - 2x = 180^\circ - \sphericalangle AFB$, which means that the quadrilateral $APBF$ is cyclic. It follows that $\sphericalangle APF = \sphericalangle ABF = 90^\circ - x$, i.e. $\sphericalangle APC + \sphericalangle APF = 180^\circ$. Thus, points C, P and F are co-linear, and FC is the radical axis of the two circles mentioned above. This line is perpendicular to the line DE determined by the centers of the two circles.



Another idea (*Marin Hristov, Andrei Pană, Ana Duguleanu, David Ghibu*):
The conclusion is equivalent to proving that $CD^2 + EF^2 = CE^2 + DF^2$.
Computing (with the law of cosines) the four lengths involved and using $\cos(90^\circ + t) = -\sin t$ reduces the previous relation to the law of sines in triangle ABC.

Remark: (*Tashi Diaconescu*)
The statement remains true if one constructs the three triangles towards the interior.

Soluţion: (*Stan Fulger*)
Construct the isosceles triangle ABK similar to BCD inside the original triangle. By a well known property, $DCEK$ is a parallelogram. Let M be the common midpoint of CK and DE. As $AK \perp AF$ and $BK \perp BF$, $AFBK$ is cyclic, inscribed into the circle of diameter FK. Let N be its circumcenter. Notice there is a spiral similarity sending N to F and D to C, triangles BCF and BDN being similar; likewise there is a spiral similarity sending N to F and E to C. These two operations have the same dilation ratio, consequently $NE = ND$, thus $MN \perp DE$. But MN is midline in triangle FKC and $FC \parallel MN \perp DE$, done.