Problema săptămânii 210
a) Fie a, b, c numere reale pozitive astfel încât $abc = 1$. Demonstraţi că

$$\frac{(a-1)(c+1)}{1+bc+c} + \frac{(b-1)(a+1)}{1+ca+a} + \frac{(c-1)(b+1)}{1+ab+b} \geq 0.$$

China Tianjin nr. 8/2013, problema 352

b) Fie a, b, c, d numere reale pozitive astfel încât $abcd = 1$. Demonstraţi că

$$\frac{(a-1)(c+1)}{1+bc+c} + \frac{(b-1)(d+1)}{1+cd+d} + \frac{(c-1)(a+1)}{1+da+a} + \frac{(d-1)(b+1)}{1+ab+b} \geq 0.$$

Orif Ibrogimov, Olimpiada Internaţională Zhautykov, 2013

Soluţii pentru a):
Soluţia 1:
Substituind $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$, unde $x, y, z > 0$, şi înmulţind cu $x + y + z$, inegalitatea devine $\sum_{cycl} \left(\frac{xz}{y} - z - x + \frac{x^2}{y} \right) \geq 0$, sau $\sum_{cycl} \frac{x^2}{y} + \sum_{cycl} \frac{xz}{y} \geq 2(x+y+z)$.
Dar $\sum_{cycl} \frac{x^2}{y} \geq x + y + z$ conform inegalităţii Cauchy-Buniakowsky-Schwarz (forma Titu Andreescu) şi $\sum_{cycl} \frac{xz}{y} \geq x + y + z$ (revine la $\sum_{cycl} x^2(y-z)^2 \geq 0$) conduc prin adunare la inegalitatea din enunţ.
Egalitatea are loc dacă $x = y = z$, adică pentru $a = b = c = 1$.

Soluţia 2:
Putem folosi următoarele două identităţi cunoscute: dacă $abc = 1$, atunci

$$\frac{1}{1+bc+c} + \frac{1}{1+ca+a} + \frac{1}{1+ab+b} = 1, \qquad (1)$$

$$\frac{bc}{1+bc+c} + \frac{ca}{1+ca+a} + \frac{ab}{1+ab+b} = 1. \qquad (2)$$

Aceste identităţi pot fi demonstrate fie cu substituţia de mai sus, fie aducând cele trei fracţii la acelaşi numitor în felul următor (vezi şi problema săptămânii 113):

$$\frac{1}{1+bc+c} + \frac{1}{1+ca+a} + \frac{1}{1+ab+b} = \frac{1}{1+bc+c} + \frac{bc}{bc(1+ca+a)} + \frac{c}{c(1+ab+b)} =$$
$$\frac{1}{1+bc+c} + \frac{bc}{1+bc+c} + \frac{c}{1+bc+c} = 1.$$

Analog se demonstrează şi inegalitatea (2).

Revenind la inegalitatea de la a), o putem rescrie echivalent după cum urmează:
$\sum\limits_{cycl}\dfrac{ac+a-c-1}{1+bc+c} \geq 0 \Leftrightarrow \sum\limits_{cycl}\left(\dfrac{ac+a-c-1}{1+bc+c}+1\right) \geq 3 \Leftrightarrow \sum\limits_{cycl}\dfrac{ac+a+bc}{1+bc+c} \geq 3 \overset{(2)}{\Leftrightarrow}$
$\sum\limits_{cycl}\dfrac{ac+a}{1+bc+c} \geq 2 \Leftrightarrow \sum\limits_{cycl}\left(\dfrac{ac+a}{1+bc+c}-a\right) \geq 2-a-b-c \Leftrightarrow -\sum\limits_{cycl}\dfrac{1}{1+bc+c} \geq$
$2-a-b-c \overset{(1)}{\Leftrightarrow} a+b+c \geq 3$, care rezultă imediat din inegalitatea mediilor şi condiţia $abc = 1$.

Soluţia 3: (*Titu Zvonaru*)
Folosim inegalitatea mediilor:
$\dfrac{(a-1)(c+1)}{1+bc+c}+\dfrac{(b-1)(a+1)}{1+ca+a}+\dfrac{(c-1)(b+1)}{1+ab+b} \geq \dfrac{1+ca+a}{1+bc+c}+\dfrac{1+ab+b}{1+ca+a}+$

$$\frac{1+bc+c}{1+ab+b}-\left(\frac{c+2}{1+bc+c}+\frac{a+2}{1+ca+a}+\frac{b+2}{1+ab+b}\right) \geq$$

$$3-\left(\frac{ab(c+2)}{ab(1+bc+c)}+\frac{b(a+2)}{b(1+ca+a)}+\frac{b+2}{1+ab+b}\right) =$$

$$3-\left(\frac{1+2ab}{1+ab+b}+\frac{ab+2b}{1+ab+b}+\frac{b+2}{1+ab+b}\right) = 3-3=0.$$

Pentru inegalitatea de la a) am primit soluţii de la: *Carol Luca Gasan, Adrian Zanca* şi *David Ghibu*.

Soluţie pentru b): (*Titu Zvonaru*)
Folosind inegalitatea Cauchy-Buniakowsky-Schwarz obţinem

$$\frac{(a-1)(c+1)}{1+bc+c}+1 = \frac{a+bc+ac}{1+bc+c} = \frac{(a+bc+ac)\left(\dfrac{1}{a}+bc+\dfrac{c}{a}\right)}{(1+bc+c)\left(\dfrac{1}{a}+bc+\dfrac{c}{a}\right)} \geq$$

$$\frac{a(1+bc+c)^2}{(1+bc+c)(1+abc+c)} = \frac{ad(1+bc+c)}{d(1+abc+c)} = \frac{ad(1+bc+c)}{d+1+cd}.$$

Scriind încă trei relaţii similare, cu inegalitatea mediilor rezultă concluzia dorită.

O altă scriere a aceleiaşi soluţii (folosind substituţii similare celor de la a)), este soluţia dată de Zarif_Ibragimov pe AoPS.

Problem of the week no. 210

a) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(a-1)(c+1)}{1+bc+c} + \frac{(b-1)(a+1)}{1+ca+a} + \frac{(c-1)(b+1)}{1+ab+b} \geq 0.$$

China Tianjin No.8 (2013), Problem 352

b) Let a, b, c, and d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{(a-1)(c+1)}{1+bc+c} + \frac{(b-1)(d+1)}{1+cd+d} + \frac{(c-1)(a+1)}{1+da+a} + \frac{(d-1)(b+1)}{1+ab+b} \geq 0.$$

Orif Ibrogimov, International Zhautykov Olympiad, 2013

Solutions to a):

Solution 1:

Substituting $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$, where $x, y, z > 0$, and multiplying by $x+y+z$, the inequality becomes $\sum_{cycl}\left(\frac{xz}{y} - z - x + \frac{x^2}{y}\right) \geq 0$, or $\sum_{cycl}\frac{x^2}{y} + \sum_{cycl}\frac{xz}{y} \geq 2(x+y+z)$. But $\sum_{cycl}\frac{x^2}{y} \geq x+y+z$ by Cauchy-Buniakowsky-Schwarz (Titu's Lemma) and $\sum_{cycl}\frac{xz}{y} \geq x+y+z$ (reduces to $\sum_{cycl} x^2(y-z)^2 \geq 0$) are both true, and thus we obtain the conclusion.

Equality holds if $x = y = z$, i.e. $a = b = c = 1$.

Solution 2:

We can use the following well-known identities: if $abc = 1$, then

$$\frac{1}{1+bc+c} + \frac{1}{1+ca+a} + \frac{1}{1+ab+b} = 1, \qquad (1)$$

$$\frac{bc}{1+bc+c} + \frac{ca}{1+ca+a} + \frac{ab}{1+ab+b} = 1. \qquad (2)$$

These identities can be proven either by the substitution above, or by astutely bringing the three fraction to a common denominator (see also problem of the week no. 113):

$$\frac{1}{1+bc+c} + \frac{1}{1+ca+a} + \frac{1}{1+ab+b} = \frac{1}{1+bc+c} + \frac{bc}{bc(1+ca+a)} + \frac{c}{c(1+ab+b)} = \frac{1}{1+bc+c} + \frac{bc}{1+bc+c} + \frac{c}{1+bc+c} = 1.$$

One proceeds similarly for the second identity.

We can rewrite equivalently our inequality as follows:

$$\sum_{cycl}\frac{ac+a-c-1}{1+bc+c}\geq 0 \Leftrightarrow \sum_{cycl}\left(\frac{ac+a-c-1}{1+bc+c}+1\right)\geq 3 \Leftrightarrow \sum_{cycl}\frac{ac+a+bc}{1+bc+c}\geq 3 \overset{(2)}{\Leftrightarrow}$$

$$\sum_{cycl}\frac{ac+a}{1+bc+c}\geq 2 \Leftrightarrow \sum_{cycl}\left(\frac{ac+a}{1+bc+c}-a\right)\geq 2-a-b-c \Leftrightarrow -\sum_{cycl}\frac{1}{1+bc+c}\geq$$

$2-a-b-c \overset{(1)}{\Leftrightarrow} a+b+c\geq 3$, which follows immediately from the AM-GM inequality.

Solution 3: (*Titu Zvonaru*)
We use the AM-GM inequality:

$$\frac{(a-1)(c+1)}{1+bc+c}+\frac{(b-1)(a+1)}{1+ca+a}+\frac{(c-1)(b+1)}{1+ab+b}\geq \frac{1+ca+a}{1+bc+c}+\frac{1+ab+b}{1+ca+a}+$$

$$\frac{1+bc+c}{1+ab+b}-\left(\frac{c+2}{1+bc+c}+\frac{a+2}{1+ca+a}+\frac{b+2}{1+ab+b}\right)\geq$$

$$3-\left(\frac{ab(c+2)}{ab(1+bc+c)}+\frac{b(a+2)}{b(1+ca+a)}+\frac{b+2}{1+ab+b}\right)=$$

$$3-\left(\frac{1+2ab}{1+ab+b}+\frac{ab+2b}{1+ab+b}+\frac{b+2}{1+ab+b}\right)=3-3=0.$$

Solution for b): (*Titu Zvonaru*)
Using the Cauchy-Buniakowsky-Schwarz inequality, we obtain

$$\frac{(a-1)(c+1)}{1+bc+c}+1=\frac{a+bc+ac}{1+bc+c}=\frac{(a+bc+ac)\left(\frac{1}{a}+bc+\frac{c}{a}\right)}{(1+bc+c)\left(\frac{1}{a}+bc+\frac{c}{a}\right)}\geq$$

$$\frac{a(1+bc+c)^2}{(1+bc+c)(1+abc+c)}=\frac{ad(1+bc+c)}{d(1+abc+c)}=\frac{ad(1+bc+c)}{d+1+cd}.$$

Adding with three more, similar, inequalities and applying AM-GM leads to the desired conclusion.

Another, equivalent, way of writing the same solution, but using substitutions similar to those used above, is the solution given by Zarif_Ibragimov on AoPS.