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CYPRUS

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JUNIOR BALKAN MATHEMATICAL OLYMPIAD

SHORTLISTED PROBLEMS WITH SOLUTIONS

LARNACA

CYPRUS

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Algebra

TAJ

Problem 1. *[A1]*

Prove the inequality

$$(a^5+a^4+a^3+a^2+a+1)(b^5+b^4+b^3+b^2+b+1)(c^5+c^4+c^3+c^2+c+1) \geq 8(a^2+a+1)(b^2+b+1)(c^2+c+1)$$

for $a, b, c \in \mathbb{R}_+$ *such that* $abc = 1$.

Solution

By factorizing we get $(a^5+a^4+a^3+a^2+a+1) = (a^3+1)(a^2+a+1)$. *We apply same thing to the other terms and simply to get* $(a^3+1)(b^3+1)(c^3+1) \geq 8$. *By* $AM \geq GM$ *we have*

$$a^3+1 \geq 2\sqrt{a^3}$$
$$b^3+1 \geq 2\sqrt{b^3}$$
$$c^3+1 \geq 2\sqrt{c^3}$$

$$(a^3+1)(b^3+1)(c^3+1) \geq 8\sqrt{a^3b^3c^3} = 8$$

Equality holds if and only if $a = b = c = 1$.

Problem 2. *[A2]* MCD

Let x, y *and* z *are positive real numbers. Prove that:*

$$\frac{x+2y}{z+2x+3y} + \frac{y+2z}{x+2y+3z} + \frac{z+2x}{y+2z+3x} \leq \frac{3}{2}$$

Solution

Let $a = \frac{x+2y}{z+2x+3y}, b = \frac{y+2z}{x+2y+3z}$ *and* $c = \frac{z+2x}{y+2z+3x}$.
We have $\frac{1}{1-a} = \frac{z+2x+3y}{x+y+z}$, $\frac{1}{1-b} = \frac{x+2y+3z}{x+y+z}$ *and* $\frac{1}{1-c} = \frac{y+2z+3x}{x+y+z}$.
It is easy to see that

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{6\,(x+y+z)}{x+y+z} = 6$$

If we use the inequality between arithmetic and harmonic mean for the positive numbers $1-a, 1-b$ *and* $1-c$ *we get*

$$\frac{1-a+1-b+1-c}{3} \geq \frac{3}{\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}} = \frac{1}{2}$$

From the last we get $a+b+c \leq \frac{3}{2}$.

Problem 3. *[A3]* MNE

Let a and b be positive real numbers. Prove that

$$\sqrt{\frac{a^2+ab+b^2}{3}} + \sqrt{ab} \leq a+b$$

Solution

First Solution. After dividing both sides by b it becomes

$$\sqrt{\frac{1}{3}\left(\left(\frac{a}{b}\right)^2+\frac{a}{b}+1\right)} + \sqrt{\frac{a}{b}} \leq \frac{a}{b}+1,$$

which by substituting $\frac{a}{b} = x$ *gives*

$$\sqrt{\frac{x^2+x+1}{3}} + \sqrt{x} \leq x+1.$$

Taking squares the above inequality transforms into

$$\frac{x^2+x+1}{3} + x + 2\cdot\sqrt{\frac{x(x^2+x+1)}{3}} \leq x^2+2x+1,$$

or equivalently,

$$2\cdot\sqrt{\frac{x(x^2+x+1)}{3}} \leq \frac{2(x^2+x+1)}{3},$$

which dividing by $\frac{2}{3}\sqrt{x^2+x+1}$ *becomes* $\sqrt{3x} \leq \sqrt{x^2+x+1}$.
Taking squares the above inequality transforms into $x^2-2x+1 \geq 0$, *which can be written as* $(x-1)^2 \geq 0$. *Hence, the required inequality is proved, and equality holds if and only if* $a=b$.

Second Solution. Applying the well known inequality $\frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$, $x \geq 0, y \geq 0$, *with* $x = \sqrt{\frac{a^2+ab+b^2}{3}}$ *and* $y = \sqrt{ab}$, *we find that*

$$\sqrt{\frac{a^2+ab+b^2}{3}} + \sqrt{ab} \leq \sqrt{2}\cdot\sqrt{\frac{a^2+ab+b^2}{3}+ab} = \sqrt{\frac{2a^2+2b^2+8ab}{3}}.$$

We will show that the right hand side of the above inequality is $\leq a+b$, *that is,*

$$\sqrt{\frac{2a^2+2b^2+8ab}{3}} \leq a+b.$$

Taking squares, after easy calculations, the above inequality reduced to $a^2 - 2ab + b^2 \geq 0$, which can be written as $(a-b)^2 \geq 0$. Hence, the required inequality is proved, and equality holds if and only if $a = b$.

Third Solution Let $\sqrt{\frac{a^2+ab+b^2}{3}} = x$ and $\sqrt{ab} = y$, then $a^2+ab+b^2 = 3x^2$ and $ab = y^2$. We have $x + y \leq a + b$. Squaring both sides we have $2x(x-y) \geq 0$ witch is true because $x \geq y \geq 0$

Problem 4. *[A4]* BUL

Let x, y, be positive numbers such that $x^3 + y^3 \leq x^2 + y^2$. Find the greatest possible value of the product xy.

Solution

Put $x + y = a > 0$, $xy = b > 0$. The given inequality writes as $a^3 - 3ab \leq a^2 - 2b$ that is equivalent to $a^3 - a^2 \leq b(3a-2)$. Note that $(x-y)^2 \geq 0$ implies $b \leq \frac{a^2}{4}$. If $a \leq \frac{2}{3}$, it follows that $b \leq \frac{1}{9}$. Now let $a > \frac{2}{3}$. From $a^3 - a^2 \leq b(3a-2)$ we get $\frac{a^3-a^2}{3a-2} \leq b$, but $b \leq \frac{a^2}{4}$ leads to $\frac{a^3-a^2}{3a-2} \leq \frac{a^2}{4} \Rightarrow a \leq 2$. By $b \leq \frac{a^2}{4}$ we get $b \leq 1$. For $x = y = 1$ we have $b = 1$, so this is the greatest value of b.

Problem 5. *[A5]* GRE

Determine positive integers a, b such that

$$a^2b^2 + 208 = 4 \cdot ([a,b] + (a,b))^2$$

where $[a,b]$ is the least common multiple and (a,b) is the greater common divisor of the positive integers a, b.

Solution

We put $[a,b] = x$ and $(a,b) = y$. From the well known identity for positive integers

$$ab = [a,b] \cdot (a,b)$$

the given equation is equivalent to

$$x^2y^2 + 208 = 4(x+y)^2 \Leftrightarrow \{2(x+y) + xy\} \cdot \{2(x+y) - xy\} = 208$$

Since the two factors $\mathrm{A} = 2(x+y) + xy$ and $\mathrm{B} = 2(x+y) - xy$ are positive integers having sum $\mathrm{A}+\mathrm{B} = 4(x+y)$ a multiple of 4 and difference $\mathrm{A}-\mathrm{B} = 2xy$ an even number, from the last equation we conclude that only the pair $(\mathrm{A}, \mathrm{B}) =$

$(52, 4)$ *is acceptable. Therefore we have*

$$\left\{\begin{array}{l} \mathrm{A} = 2(x+y) + xy = 52 \\ \mathrm{B} = 2(x+y) - xy = 4 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} x+y = 14 \\ xy = 24 \end{array}\right\} \Leftrightarrow (x,y) = (12,2) \text{ or } (x,y) = (2,12)$$

Since we have $x \geq y$ *we conclude that* $(x, y) = (12, 2)$, *that is*

$$[a,b] = 12 \quad \text{and} \quad (a,b) = 2$$

Thus we have that $a = 2a_1$, $b = 2b_1$, *where* a_1, b_1 *are positive integers with* $(a_1, b_1) = 1$ *and then we get*

$$ab = xy \Rightarrow 4a_1b_1 = 12 \cdot 2 = 24 \Rightarrow a_1b_1 = 6.$$

Since $(a_1, b_1) = 1$ *we conclude that*

$$a_1 = 3,\ b_1 = 2 \text{ or } a_1 = 2,\ b_1 = 3.$$

Hence we have the solutions $a = 6,\ b = 4$ or $a = 4,\ b = 6$.

Problem 6. *[A6]* TAJ

Let $x_i > 1,\quad \forall i \in \{1, 2, 3, \ldots, 2011\}$. *Prove the inequality*

$$\frac{(x_1)^2}{x_2 - 1} + \frac{(x_2)^2}{x_3 - 1} + \frac{(x_3)^2}{x_4 - 1} + \cdots + \frac{(x_{2010})^2}{x_{2011} - 1} + \frac{(x_{2011})^2}{x_1 - 1} \geq 8044$$

When does equality hold?

Solution

Realize that $(x_i - 2)^2 \geq 0 \Leftrightarrow (x_i)^2 \geq 4(x_i - 1)$. *So we get,*

$$\frac{(x_1)^2}{x_2 - 1} + \frac{(x_2)^2}{x_3 - 1} + \frac{(x_3)^2}{x_4 - 1} + \cdots + \frac{(x_{2010})^2}{x_{2011} - 1} + \frac{(x_{2011})^2}{x_1 - 1} \geq$$

$$\underbrace{4\left(\frac{x_1 - 1}{x_2 - 1} + \frac{x_2 - 1}{x_3 - 1} + \cdots + \frac{x_{2010} - 1}{x_{2011} - 1} + \frac{x_{2011} - 1}{x_1 - 1}\right)}_{denote A}$$

By $AM \geq GM$ *we have*

$$\frac{x_1 - 1}{x_2 - 1} + \frac{x_2 - 1}{x_3 - 1} + \cdots + \frac{x_{2010} - 1}{x_{2011} - 1} + \frac{x_{2011} - 1}{x_1 - 1} \geq$$

$$2011 \cdot \sqrt[2011]{\frac{x_1 - 1}{x_2 - 1} \cdot \frac{x_2 - 1}{x_3 - 1} \cdots \frac{x_{2010} - 1}{x_{2011} - 1} \cdot \frac{x_{2011} - 1}{x_1 - 1}}$$

so $A \geq 2011$

Equality holds when $x_1 = x_2 = \cdots = x_{2011}$ *and* $(x_i - 2)^2 = 0$

So equality holds when $x_i = 2, \forall i \in \{1, 2, 3, \ldots, 2011\}$

Problem 7. *[A7]* TAJ

Let $a, b, c \in \mathbb{R}_+$ and $abc = 1$. Prove the inequality

$$\frac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1}+\frac{2b^2+\frac{1}{b}}{c+\frac{1}{b}+1}+\frac{2c^2+\frac{1}{c}}{a+\frac{1}{c}+1}\geq 3$$

Solution

By AM-GM inequality we have $2a^2+\frac{1}{a}=a^2+a^2+\frac{1}{a}\geq 3\sqrt[3]{a^3}=3a$ *applying same process to the other numeraters we get*

$$\frac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1}+\frac{2b^2+\frac{1}{b}}{c+\frac{1}{b}+1}+\frac{2c^2+\frac{1}{c}}{a+\frac{1}{c}+1}\geq\frac{3a}{b+\frac{1}{a}+1}+\frac{3b}{c+\frac{1}{b}+1}+\frac{3c}{a+\frac{1}{c}+1}\overset{?}{\geq}3$$

From here we need to prove

$$\frac{a}{b+\frac{1}{a}+1}+\frac{b}{c+\frac{1}{b}+1}+\frac{c}{a+\frac{1}{c}+1}\overset{?}{\geq}1$$

Since $\frac{1}{a}=bc,\quad \frac{1}{b}=ac,\quad \frac{1}{c}=bc$ *we get*

$$\underbrace{\frac{a}{b+bc+1}+\frac{b}{c+ca+1}+\frac{c}{a+ab+1}}_{\text{denote as }A}\overset{?}{\geq}1$$

In order to prove that A is bigger than 1. We try to show the following inequality

$$\underbrace{\frac{a}{b+bc+1}+\frac{b}{c+ca+1}+\frac{c}{a+ab+1}}_{\text{denote as }A}[a(b+bc+1)+b(c+ca+1)+c(a+ab+1)]$$
$$\overset{?}{\geq}(a+b+c)^2$$

Here if we show that

$$(a(b+bc+1)+b(c+ca+1)+c(a+ab+1))\leq(a+b+c)^2$$

it implies that A is bigger than 1.

So we just need to prove

$$(ab+1+a+bc+1+b+ca+1+c)\overset{?}{\leq}(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$$

or

$$3+a+b+c\overset{?}{\leq}a^2+b^2+c^2+ab+ac+bc$$

Here we can show that $3\leq ab+ac+bc$ *by AM-GM inequality and we can easily show that* $a+b+c\leq a^2+b^2+c^2$. *This finishes the proof.*

Problem 8. *[A8]* MNE

Decipher the equality

$$(\overline{LARN} - \overline{ACA}) : (\overline{CYP} + \overline{RUS}) = C^{Y^P} \cdot R^{U^S}$$

where different symbols correspond to the different digits, and the equal symbols correspond to the equal digits. It is also suposed that all these digits are different from 0.

Solution

Denote $x = \overline{LARN} - \overline{ACA}$, $y = \overline{CYP} + \overline{RUS}$ and $z = C^{Y^P} \cdot R^{U^S}$. It is obvious that $1823 - 898 \leq x \leq 9287 - 121$, $123 + 456 \leq y \leq 987 + 654$, that is $925 \leq x \leq 9075$ and $579 \leq y \leq 1641$, whence it follows that $\frac{925}{1641} \leq \frac{x}{y} \leq \frac{9075}{579}$, or $0,563... \leq \frac{x}{y} \leq 15,673....$ Since $\frac{x}{y} = z = C^{Y^P} \cdot R^{U^S}$ is an integer, it follows that this ilso a number $\frac{x}{y}$, and so, the previous inequality yields $1 \leq \frac{x}{y} \leq 15$. Hence, must be $1 \leq z \leq 15$, that is $1 \leq C^{Y^P} \cdot R^{U^S} \leq 15$. Hence, both values C^{Y^P} and R^{U^S} are ≤ 15. From this and the fact that $2^{2^2} = 16$ it follows that at least one of the symbols in the expression C^{Y^P} and at least one of the symbols in the expression R^{U^S} correspond to the digit 1. This is impossible because of the assumption that all the symbols in the set $\{C, Y, P, R, U, S\}$ correspond to the different digits.

Problem 9. *[A9]* ROM

Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$\sum_{k=1}^{n-1} min(x_k, x_{k+1}) = min(x_1, x_n).$$

Show that

$$\sum_{k=2}^{n-1} x_k \geq 0$$

Solution

Since $\min(a, b) = \frac{1}{2}(a + b - |a - b|)$, we have

$$\sum_{k=1}^{n-1} \frac{1}{2}(x_k - x_{k+1} - |x_k - x_{k+1}|) = \frac{1}{2}(x_1 + x_n - |x_1 - x_n|) \Leftrightarrow \ldots$$

$$2(x_2 + \ldots + x_{n-1}) + |x_1 - x_n| = |x_1 - x_2| + \ldots + |x_{n-1} - x_n|.$$

As $|x_1 - x_2| + \cdots + |x_{n-1-x_n}| \geq |x_1 - x_2 + \cdots + x_{n-1} - x_n| = |x_1 - x_n|$, we obtain the desired conclusion.

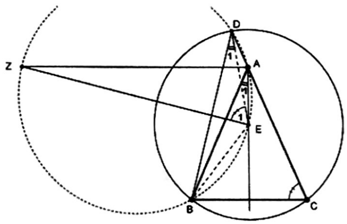
Geometry

Problem 1. *[G1]* GRE

Let ABC be an isosceles triangle with $AB = AC$. On the extension of the side CA we consider the point D such that $AD < AC$. The perpendicular bisector of the line segment BD meets the internal and external bisector of the angle $\angle A$ at the points E and Z, respectively. Prove that the points $A, \cancel{B}$ E$, D, Z$ are cocyclic.

Solution

First Solution In the triangle the line bisects the angle and the line is the perpendicular bisector of the side . Hence belongs to the circumcircle of the triangle . Therefore the points are cocyclic.



In the triangle BCD, AE and ZE are perpendicular bisectors of the sides BC and BD respectively. Hence, E is the circumcentre of the triangle BCD and therefore $\hat{E}_1 = \hat{C} = \hat{B}$.

Since $BD \perp ZE$, we conclude that:

$$\hat{D}_1 = 90^o - \hat{E}_1 = 90 - \hat{B} = \hat{A}_1$$

Hence the quadrilateral $AEBD$ is cyclic, that is the points A, B, D, E are cocyclic. Therefore, since A, B, D, Z are also cocyclic we conclude that $AEZD$ is cyclic.

Second solution

We consider the point T symmetric of B with respect to the axis AZ. Since AE

and BT are both perpendiculars to AZ, they are parallel and so:

$$\angle EAC = \angle BTA \tag{1}$$

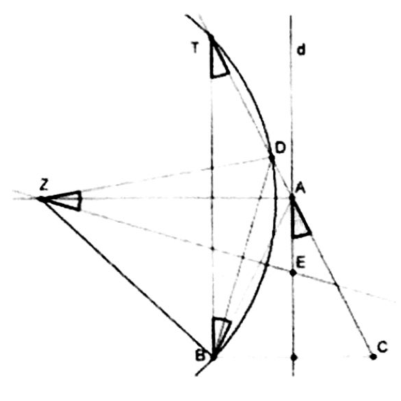
Also, because of the symmetry of B, T with respect to the axis AZ, we have

$$\angle BTA = \angle TBA \tag{2}$$

Since $ZB = ZT = ZD$, the point Z is the center of the circle passing through the points B, D, T. Therefore we have

$$\angle BTA = \frac{1}{2}\angle BZD = \angle EZD \tag{3}$$

From relations (1), (2) and (3) we conclude that $\angle EAC = \angle EZD$, which gives that the quadrilateral is inscribable, that is the points the points A, D, Z, E are cocyclic.

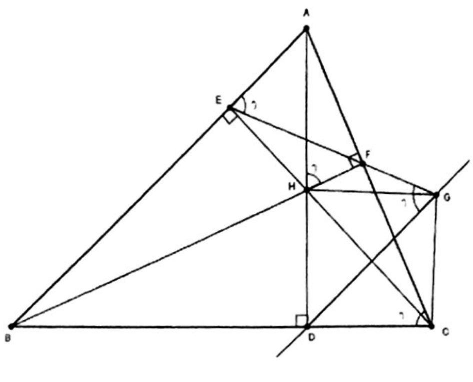


Problem 2. *[G2]* SER

Let AD, BF and CE be the altitudes of triangle $\triangle ABC$. A line passes through D and parallel to AB intersects the line EF at the point G. If H is the orthocenter of $\triangle ABC$ find the angle $\angle CGH$.

Solution

Quadrilateral $BCFE$ is cyclic because $\angle BEC = \angle BFC = 90°$ and thus $\angle AEF = \angle ACB = \gamma$. Then $\angle DGF = \angle AEF = \gamma$ (angles with parallel rays) and points D, C, F and G are concyclic. Also $\angle AHF = \angle DCF = \gamma$ (angles with orthogonal rays) and points D, C, F and H are concyclic. Thus $\angle CGH = \angle HDC = 90°$.



Problem 3. *[G3]* MNE

Let ABC be a triangle in which BL is the angle bisector of $\angle ABC$, AH is the altitude of ABC and M is the midpoint of the side AB ($L \in AC$, $H \in BC$ and $M \in AB$). It is known that the midpoints of the segments BL and MH coincide. Determine the internal angles of ABC.

Solution

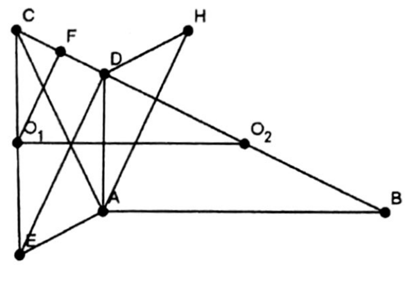
Let N be the intersection of the segments BL and MH. Because by the assumption N is a midpoint of both segments BL and MH, it follows that $BMLH$ is a parallelogram. This implies that $ML||BC$ and $LH||AB$, and hence, the angle bisector BL and the altitude AH are also the medians of ABC. This shows that ABC is an equilateral triangle with all internal angles measuring 60^0.

Problem 4. *[G4]* BUL

Point D lies on the side BC of triangle ABC. The circumcenters of triangles ADC and BAD are O_1 and O_2, respectively, and $O_1O_2||AB$. The orthocenter of triangle ADC is H and $AH = O_1O_2$. Find the angles of triangle ABC, if $\angle C : \angle B = 3 : 2$.

Solution

As O_1 and O_2 lie on the perpendicular bisector of AD, we have $O_1O_2 \perp AD$. So $\angle DAB = 90^0$ and is obtuse. Let F be the midpoint of CD and CE be a diameter of the circumcircle of triangle ADC. Then $ED \perp CD$ and $EA \perp CA$, so $ED||AH$ and $EA||DH$ and hence $EAHD$ is a parallelogram. Therefore $O_1O_2 = AH = ED = 2O_1F$ and so $\angle O_1O_2F = \angle ABC = 30^0$. Now as $\angle C : \angle B = 3 : 2$, we get $\angle C = 45^0$, $\angle A = 105^0$.



Problem 5. *[G5]* MOL

Inside the square $ABCD$ the equilateral triangle ABE is constructed. Let M be an interior point of the triangle ABE, such that $MB = \sqrt{2}$, $MC = \sqrt{6}$, $MD = \sqrt{5}$ and $ME = \sqrt{3}$. Find the area of the square $ABCD$.

Solution

First Solution. For every point T, such that the points T, A, C and T, B, D are noncoliniar, we have the following assertion: $TA^2 + TC^2 = TB^2 + TD^2$. Let $AC \cap BD = \{O\}$ and consider the triangles ACT and BDT. In each of them the segment TO is a median. By applying twice the median formulae we obtain

$$4 \cdot TO^2 = 2(TA^2 + TC^2) - AC^2, \quad 4 \cdot TO^2 = 2(TB^2 + TD^2) - BD^2.$$

Because $AC = BD$, from the last equalities we have $TA^2 + TC^2 = TB^2 + TD^2$.

Let $T = M$. So, $MA^2 + MC^2 = MB^2 + MD^2$. From the given conditions we obtain the equality $MA = 1$.

Let N be a point, situated outside of the square $ABCD$, such that $m(\angle NAB) = m(\angle MAE)$ and $m(\angle NBA) = m(\angle MEA)$. Because $AB = AE$, the triangles AME and ANB are congruent and $AM = AN = 1$, $ME = BN = \sqrt{3}$. (see fig. 1).

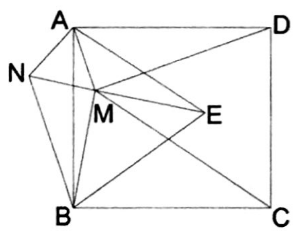


fig. 1

From the equalities $m(\angle MAE) + m(\angle MAB) = 60^\circ = m(\angle NAB) + m(\angle MAB) = m(\angle NAM)$, it follows that the triangle AMN is equilateral and $MN = 1$. The triangle BMN is rightangled, because $BM^2 + MN^2 = BN^2$. So, $m(\angle BMA) =$

$m(\angle BMN) + m(\angle AMN) = 150^\circ$. *Let P be orthogonal projection of the point A on the straightline BM (see fig.2)*

fig.2

The triangle PAM is rightangled with $m(\angle APM) = 90^\circ$, $m(\angle AMP) = 30^\circ$ and $MA = 1$. So, $AP = \frac{1}{2}$ and $PM = \frac{\sqrt{3}}{2}$. By applying Pythagoras theorem for the rightangled triangle APB we obtain

$$AB^2 = AP^2 + BP^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2} + \sqrt{2}\right)^2 = 3 + \sqrt{6}.$$

So, the area of the square $ABCD$ is equal to $3 + \sqrt{6}$ s.u.

Second Solution. Let the K, F, H, Z the the projection of point M on the sides of the square. Then by Pythagorean theorem we can prove that $MA^2 + MC^2 = MB^2 + MD^2$. From the given condition we obtain $MA = 1$.
With center A and angle 60^0 we rotate the triangle AME so we construct the triangle ANB. After we continue as solution 1.

Problem 6. *[G6]* SER

Let $ABCD$ be a convex quadrilateral, E and F points on sides AB and CD respectively, such that $AB : AE = CD : DF = n$. Denote S area of quadrilateral $AEFD$. Prove

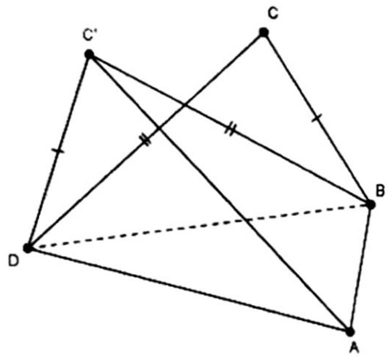
$$S \leq \frac{AB \cdot CD + n(n-1)DA^2 + n^2 DA \cdot BC}{2n^2}.$$

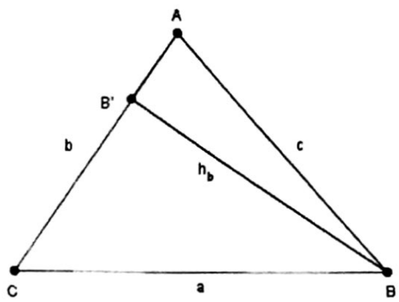
Solution

We will start with next

Lemma. Let $ABCD$ be a quadrilateral and S its area. Then

$$S \leq \frac{AB \cdot CD + BC \cdot DA}{2}.$$





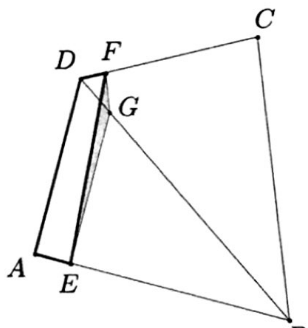
Proof. Construct point C' such that $BC' = CD$ and $DC' = BC$. Obviously, $\triangle BCD$ is congruent to $\triangle BC'D$ and the areas of quadrilaterals $ABCD$ and $ABC'D$ are equal. Area of triangle is less or equal to a half of product of two sides. Then

$$S = S_{\triangle ADC'} + S_{\triangle ABC'} \le \frac{AD \cdot DC'}{2} + \frac{AB \cdot BC'}{2} = \frac{AB \cdot CD + BC \cdot DA}{2}.$$

Applying lemma on quadrilateral $AEFD$ we get

$$S \le \frac{AE \cdot DF + DA \cdot EF}{2} =$$

$$= \frac{AB \cdot CD + n^2 DA \cdot EF}{2n^2}.$$



Let G be a point on diagonal BD such that $DB : DG = n$. From Thales's theorem we get $GE = \frac{(n-1)AD}{n}$ and $GF = \frac{BC}{n}$. Applying inequality of triangle on $\triangle EGF$ we get

$$EF \le EG + GF = \frac{(n-1)AD + BC}{n}.$$

Now, we get

$$S \le \frac{AB \cdot CD + n^2 DA \cdot EF}{2n^2} \le \frac{AB \cdot CD + n(n-1)DA^2 + nDA \cdot BC}{2n^2}.$$

Number Theory

Problem 1. *[NT1]* SER

Solve equation

$$1005^x + 2011^y = 1006^z$$

in the set of natural numbers.

Solution

Note that $1005 \equiv 2011 \equiv -1 \pmod{1006}$, *so* $1006 \mid 1005^x + 2011^y$ *follows to, from numbers* x *and* y, *exactly one is odd and one is even.*

Consequence of this fact is that 8 *can not divide* $1005^x + 2011^y$, *because, if* x *is even,* y *is odd, then* $1005^x + 2011^y \equiv 1 + 3 = 4 \pmod{8}$, *and if* x *is odd,* y *is even, then* $1005^x + 2011^y \equiv 5 + 1 = 6 \pmod{8}$. *Now, we conclude* $z \leq 2$. *We know* $2011 > 1006$, *so* $z > 1$. *Only possibility for* z *is* $z = 2$, *and this follows to* $y = 1$, *and* $x = 2$, *which is unique solution of this equation.*

Solution $(x, y, z) = (2, 1, 2)$.

Problem 2. *[NT2]* MOL

Find all prime numbers p *for which the equation*

$$x(y^2 - p) + y(x^2 - p) = 5p$$

has the solutions (x, y) *in natural numbers.*

Solution

Given equation is equivalent to $(x + y)(xy - p) = 5p$. *We consider the following cases:*

1. *Let* $x + y = 1$ *and* $xy = 6p$. *For prime* $p \geq 2$ *the equation* $x^2 - x + 6p = 0$ *has no solutions.*

2. *Let* $x + y = 5$ *and* $xy = 2p$. *For prime* $p \geq 2$ *the equation* $x^2 - 5x + 2p = 0$ *has the discriminant* $\Delta = 25 - 8p$. *The inequality* $25 - 8p \geq 0$ *implies* $p \in \{2, 3\}$.

For $p = 2$ we obtain the solutions $(1, 4)$ and $(4, 1)$.

For $p = 3$ we obtain the solutions $(2, 3)$ and $(3, 2)$.

3. Let $x+y = p$ and $xy = p+5$. For prime $p \geq 2$ the equation $x^2-px+p+5 = 0$ has the discriminant $\Delta = p^2 - 4p - 20$. The inequality $p^2 - 4p - 20 \geq 0$ implies $p \geq 7$.

Let $p^2 - 4p - 20 = q^2$ with $1 \leq q < p$. We obtain the equation $(p-2)^2 - q^2 = 24$ which is equivalent to $(p + q - 2)(p - q - 2) = 24$. It follows that both numbers $p + q - 2$ and $p - q - 2$ shall be even. We have two subcases:

a) $p + q - 2 = 12$ and $p - q - 2 = 2$. We have p=9, which is no prime.

b) $p + q - 2 = 6$ and $p - q - 2 = 4$. We obtain $p = 7$ and $q = 1$. The equation has the solutions $(3, 4)$ and $(4, 3)$.

4. Let $x + y = 5p$ and $xy = p + 1$. It follows that $p \notin \mathbb{N}$.

So, the equation has natural solutions only for $p \in \{2, 3, 7\}$.

Problem 3. *[NT3]* BUL

Find all positive integers n such that the equation $y^2+xy+3x = n\left(x^2 + xy + 3y\right)$ has a solution in positive integers x, y.

Solution

Clearly for $n = 1$ any pair x, y such that $x = y$ is a solution. Now let $n > 1$, so $x \neq y$. $0 < n - 1 = \frac{y^2+xy+3x}{x^2+xy+3y} - 1 = \frac{(x+y-3)(y-x)}{x^2+xy+3y}$. As $x + y \geq 3$, we conclude that $x + y > 3$ and $y > x$. Let $d = GCD\left(x + y - 3, x^2 + xy + 3y\right)$. Then d divides $x^2 + xy + 3y - x\,(x + y - 3) = 3\,(x + y)$. Then d also divides $3\,(x + y) - 3\,(x + y - 3) = 9$, hence $d = 1$, $d = 3$ or $d = 9$. As $n - 1 = \frac{\frac{x+y-3}{d}(y-x)}{\frac{x^2+xy+3y}{d}}$ and the positive integers $\frac{x+y-3}{d}$ and $\frac{x^2+xy+3y}{d}$ are relatively prime, it follows that $\frac{x^2+xy+3y}{d}$ must divide $y - x$, which leads to $x^2 + xy + 3y \leq dy - dx \Leftrightarrow$ and so $x^2+dx \leq (d - 3 - x)\,y$. Therefore $d = 9$ and $x < 6$. As now $x+y-3$ is a multiple of 9, let $x + y - 3 = 9k$, $k \in N$, so $y = 9k + 3 - x$. Hence $n - 1 = \frac{k(9k+3-2x)}{k(x+3)+1}$ and as k and $k\,(x + 3) + 1$ are relatively prime, the number $t = \frac{9k+3-2x}{k(x+3)+1}$ must be integer for some positive integer $x < 6$. It remains to consider these values of x.

- For $x = 1$ one has $t = \frac{9k+1}{4k+1} < 3$ and $t > 1$, hence $t = 2$, $k = 1$, $y = 11$ and $n = 3$.

- For $x = 2$ one has $t = \frac{9k-1}{5k+1}$ and $1 < t < 2$, so there are no solutions here.

- For $x = 3$ one has $t = \frac{9k-3}{6k+1} < 2$ and $t = 1$; there are no solutions here.

- For $x = 4$ one has $t = \frac{9k-5}{7k+1} < 2$ and $t = 1$ leads to $k = 3, \quad y = 26$ and $n = 4$.

- For $x = 5$ one has $t = \frac{9k-7}{8k+1} < 2$ and $t = 1$ leads to $k = 8, \quad y = 70$ and $n = 9$.

Answer: $n = 1, n = 3, n = 4$ and $n = 9$.

Problem 4. *[NT4]* GRE

Find prime positive integers p, q such that $2p^3 - q^2 = 2(p+q)^2$.

Solution

First Solution The given equation is written

$$2p^2(p-1) = q(3q+4p) \qquad (1)$$

From (1) we get that $p \mid q(3q+4p)$ and since p must be prime we get that

$$p \mid q \qquad or \qquad p \mid 3q+4p$$

- If $p \mid q$ then $p = q$, because p, q are primes. Then equation (1) becomes $2p^3 - 9p^2 = 0$ which has not prime solutions.

- If $p \mid 3q+4p$, then $p \mid 3q$. Since we have rejected the case we finally conclude that $p \mid 3$ and then $p = 3$ (since p is prime).

For $p = 3$, equation (1) gives that $q^2+4q-12 = 0 \Leftrightarrow q = 2$ or $q = -6$ (it is rejected)

Hence we have the solution $(p, q) = (3, 2)$.

Second Solution The given equation becomes

$$p^3 = \frac{2p^2 + 4pq + 3q^2}{2}$$

Since p positive integer then $2 \mid 3q^2$. Since the numbers 2 and 3 are relatively prime, we have $2 \mid q^2$, thus $q = 2$. So the initial equation is transformed to $p^3 - p^2 - 4p - 6 = 0$. The only natural solution is $p = 3$. Hence we have the solution $(p, q) = (3, 2)$

Problem 5. *[NT5]* BUL

Find the least positive integer such that the sum of its digits is 2011 and the product of its digits is a power of 6.

Solution

First Solution. *Denote this number by N. Then N cannot contain the digits 0, 5 or 7; also, its digits must be written in non-decreasing order. Note that:*

1. *N cannot have more than 4 digits "1", otherwise the substitution of the digits* $11111 \to 23$ *would lead to a smaller number with the required properties.*

2. *N cannot have more than one "2" due to the substitution* $22 \to 4$.

3. *N cannot have more than two "4". Indeed, suppose that there are at least three "4". Then the product of digits of N is at least* 6^6 *. In this case N cannot contain "9" due to the substitution* $\{44;9\} \to \{36;8\}$ *, nor "6" due to the substitution* $4446 \to 1188$. *But then it must contain at least six "3", which is impossible due to the substitution* $33344 \to 89$.

4. *N cannot have more than six "6" due to the substitution* $6666666 \to 1248999$.

5. *As* $2011 = 9 \cdot 223 + 4$, *N has at least 224 digits; since at most 4 of them are "1", at least 220 digits of N have a prime divisor 2 or 3. Suppose that among these 220 digits there are not more than 54 even digits; then the odd ones are at least* $220 - 54 = 166$ *and the power of 3 in N is at least 166, while the power of 2 in N is at most* $3 \cdot 54 = 162$, *contradicting the fact that the product of digits of N is a power of 6. Thus N has at least 55 even digits, including at most one "2", at most two "4" and at most six "6", and hence at least* $55 - (1 + 2 + 6) = 46$ *digits «8». Thus the product of digits of N is at least* $(6^3)^{46} = 6^{138}$.

6. *N cannot have more than one "3" due to the substitution* $\{33;8\} \to \{26;6\}$.

7. *By 6), the digits "9" in N are at least* $[138 - (1 + 6)] : 2$, *i.e. at least 66. But then N cannot have more than one "4" due to the mentioned substitution* $\{44;9\} \to \{36;8\}$.
   Thus N can have at most: four "1" one "2", one "3", one "4" and six "6", so if S *is the sum of all digits of N, other than "8" and "9", then* $0 \le S \le 49$. *Let N have exactly* x *digits "8" and* y *digits "9". After ignoring the digits "6", the powers of 2 and 3 in the product of the remaining digits are equal.*

The power of 2 is $3x, 3x+1, 3x+2$ or $3x+3$, while the power of 3 is $2y$ or $2y+1$, so $-3 \leq 3x-2y \leq 1$. Also, $S+8x+9y=2011$ implies $1962 \leq 8x+9y \leq 2011$. We express $x+y=\frac{5}{43}(8x+9y)+\frac{1}{43}(3x-2y)$, which due to the mentioned bounds leads to $229 \leq x+y \leq 233$. Also note that $2(x+y)+3x-2y=5x$ is a multiple of 5. We now consider the possible values of $x+y$.

If $x+y=229 \equiv 4 \pmod 5$, then $3x-2y \equiv 2 \pmod 5$, whence $3x-2y=-3$. By $x+y=229$ and $3x-2y=-3$ we get $x=91, y=138$, $8x+9y=1970$, $S=41$, so N has at least $229+7=236$ digits.

If $x+y=230$, as above we get $3x-2y=0$, $x=92, y=138, S=33$ and N has at least $230+6=236$ digits.

If $x+y=231$, we get $3x-2y=-2$, $x=92, y=139, S=24$. Here N can have less than 236 digits only if it has only "8"s, "9"s and four "6", but then one cannot have $3x-2y=-2$. Thus here again N has at least 236 digits.

If $x+y=232$, we get $3x-2y=1$, $x=93, y=139, S=16$. The identity $3x-2y=1$ shows that N has one "3"and no "2"s or "4"s. Then the "6"s and "1"s in N must add up to 13, so there are at least three more digits, and here again N has at least 236 digits.

If $x+y=233$, we get $3x-2y=-1$, $x=93, y=140, S=7$. As $S=7$, N has at least two digits different from "8" and "9". By $3x-2y=-1$ and $S=7$, this is possible only of these digits are a "3" and a "4 (the possibility "1" and "6" violates $3x-2y=-1$). So N has exactly 235 digits: a "3", a "4", 93 "8"s and 140 "9" and $N=34\underbrace{88\ldots8}_{93}\underbrace{99\ldots9}_{140}$.

Remark: *It is wrong to argue that in order to make N minimal, one must take as many "9"s as possible; this leads to $N=1\underbrace{44\ldots4}_{148}8888\underbrace{99\ldots9}_{154}$ that has 307 digits and hence is much larger.*

<u>Second Solution</u>. Denote this number by N. Then N cannot contain the digits 0, 5 or 7; also, its digits must be written in non-decreasing order. Let N have x_1 ones, x_2 twos, x_3 threes, x_4 fours, x_6 sixes, x_8 eights and x_9 nines, so

$$x_1+2x_2+3x_3+4x_4+6x_6+8x_8+9x_9=2011 \qquad (1)$$

The product of digits is a power of 6 exactly when $x_2+2x_4+x_6+3x_8=$

$x_3 + x_6 + 2x_9$, i.e.

$$x_2 - x_3 + 2x_4 + 3x_8 - 2x_9 = 0 \qquad (2)$$

Denote by S the number of digits of N ($x_1+x_2+x_3+x_4+x_6+x_8+x_9 = S$). In order to make the coefficients of x_8 and x_9 equal, we multiply (1) *by 5 and add* (2)*. We get* $43x_9 + 43x_8 + 30x_6 + 22x_4 + 14x_3 + 11x_2 + 5x_1 = 10055 \Leftrightarrow 43S = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$. *Then* $10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$ *is a multiple of 43 not less than 10055. The least such number is 10062, but* $10062 = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$ *means that among* $x_1, ..., x_6$ *there is at least one positive, so* $10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1 \geq 10055 + 13 = 10068$. *The next multiple of 43 is* $235 \cdot 43 = 10105$ *and from* $10105 = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$ *we get* $50 = 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$. *By writing it as* $13x_6 + 21(x_4 - 1) + 29(x_3 - 1) + 32x_2 + 38x_1 = 0$ *we see that the only possibility is* $x_1 = x_2 = x_6 = 0$, $x_3 = x_4 = 1$. *Then* $S = 235$, $x_8 = 93, x_9 = 140$. *As S is strictly minimal, so N is* $N = 34\underbrace{88\ldots8}_{93}\underbrace{99\ldots9}_{140}$.

Combinatorics

Problem 1. *[C1]* MME

Inside of a square whose side length is 1 there are a few circles such that the sum of their circumferences is equal to 10. Show that there exists a line that meets at least four of these circles.

Solution

Find projections of all given circle on one of sides of a square. A projection of each circle is a segment whose length is equal to the length of a diameter of this circle. Since the sum of lengths of all circles' diameters is equal $\frac{10}{\pi}$, *it follows that a sum of lengths of all mentioned projections is equal* $\frac{10}{\pi} > 3$. *Because of the side length of a square is 1, we conclude that at least one point, assumed a point* A, *on the projection side of a square is covered with at least four of these projections. Hence, a perpendicular line to the projection side of a square passing through the point* A *meets at least four of given circles, and so this is a line with the desired property.*

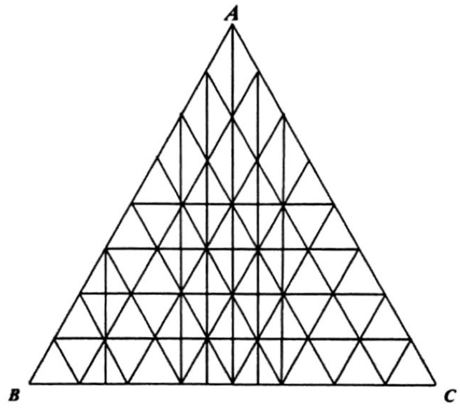
Problem 2. *[C2]* MED

Can we divide an equilateral triangle into 2011 small triangles using 122 straight lines? (there should be 2011 triangles that are not themselves divided into smaller parts and there should be no polygons which are not triangles)

Solution

For each of the sides of the triangle, we draw 37 *equidistant, parallel lines to it, (s.t. the distance between two neighboring lines is* $\frac{1}{38}$ *of the corresponding height). In this way we get* $38^2 = 1444$ *triangles. Then we delete 11 lines which are closest to the vertex A and are parallel to side a and we draw* 21 *lines perpendicular to* a, *the first starting from vertex* A *and* 10 *on each of the two sides, distributed symmetrically, as in the picture. In this way we get* $26 \cdot 21 + 10 = 556$

new triangles. Therefore we obtain a total of 2000 *triangles, and we have used* $37 + 37 + 26 + 21 = 121$ *lines. We draw the last line to be perpendicular to a, starting from the* $12-th$ *point on the side c, starting from B (including B). In this way we obtain the required division.*



Problem 3. *[C3]* SER

We can change natural number n *in these three ways:*

1^{st} *If the number* n *has at least two digits, we erase last digit and subtract that digit from remaining number (for example, from number* 123*, we are getting number* $12 - 3 = 9$*);*

2^{nd} *We can change order digits in opposite order, if the last digit is not* 0 *(for example, from number* 123*, we are getting* 321*);*

3^{rd} *We can multiply the number* n *by any number from set* $\{1, 2, 3, \dots, 2010\}$.

Can we get the number 21062011*, from the number* 1012011 *?*

Solution

Answer is NO. We will prove that if the starting number n *is divisible by* 11*, then all numbers, which we can get from* n*, are divisible by* 11.

In 1^{st}*, we get the number* $m = a - b = 11a - n$*, from the number* $n = 10a + b$ *and if* $11 \mid n$*, then* $11 \mid m$.

We know that number is divisible by 11 *if and only if difference of sum of digits on even places and sum of digits on odd places is divisible by* 11*, so when we use* 2^{nd}*, from number, which is divisible by* 11*, we get the number which is also divisible by* 11.

When we use 3^{rd}*, the number we get, obvious, remains divisible by* 11.

So, because number 1012011 *is divisible by* 11*, but* 21062011 *is not, we can not* *get this number with this changes.*

Problem 4. *[C4]* BUL

In a group of n people, each one had a different ball. They performed a sequence of swaps; in each swap, two people swapped the ball they currently had. Each pair of people performed at least one swap. At the end each person had the ball he/she had at the start. Find the least possible number of swaps, if:

a) $n = 5$ *b)* $n = 6$.

Solution

We will denote the people by A, B, C, ... and their initial balls by the corresponding small letters. Thus the initial state is Aa, Bb, Cc, Dd, Ee (, Ff). A swap is denoted by the (capital) letters of the people involved. Two people are called adjacent if their letters are adjacent.

a) *Five people form* $\frac{5\cdot 4}{2} = 10$ *pairs, so at least 10 swaps are needed. In fact 10 swaps do suffice:*

Swap AB, then BC, then CA; the state is now Aa, Bc, Cb, Dd, Ee.

Swap AD, then DE, then EA; the state is now Aa, Bc, Cb, De, Ed.

Swap BE, then CD; the state is now Aa, Bd, Ce, Db, Ec.

Swap BD, then CE; the state is Aa, Bb, Cc, Dd, Ee and all requirements are fulfilled.

Answer: 10.

b) *Six people form 6.5/2=15 pairs, so at least 15 swaps are needed. We will prove that the final number of swaps must be even. Call a pair of balls "inverted", if the ball with the former letter is in the person with the latter one. Let T denote the total number of inverted pairs; at the start* $T = 0$. *A swap performed by adjacent people changes T by 1. Any swap is equivalent to an odd number of swaps performed by adjacent people, so it changes the parity of T. Since at the end T = 0, the total number of swaps must be even. Thus at least 16 swaps are needed. In fact 16 swaps do suffice:*

Swap AB, then BC, then CA; the state is now Aa, Bc, Cb, Dd, Ee, Ff.

Swap AD, then DE, then EA; the state is now Aa, Bc, Cb, De, Ed, Ff.

Swap FB, then BE, then EF; the state is now Aa, Bd, Cb, De, Ec, Ff.

Swap FC, then CD, then DF; the state is now Aa, Bd, Ce, Db, Ec, Ff.

Swap BD, then CE, then twice AF; now all requirements are fulfilled.

Answer: 16.

Problem 5. *[C5]* SER

Set S, subset of a set of natural numbers, is called good, if for each element $x \in S$, x does not divide the sum of remaining numbers in S. Find the maximal possible number of elements of one good set, which is subset of the set $A = \{1, 2, \dots, 63\}$.

Solution

Let set B be the good subset of A, which is having maximum number of elements. We conclude that number 1 does not belong to B, because 1 divides all natural numbers. So, maximum number elements of this set is less or equal 62.

Based on the properties of divisibility, we know that x divide (does not divide) sum of remaining numbers if and only if x divide (does not divide) sum of all numbers in the set B. Now, we can prove that set B has less than 62 numbers because, if B has exactly 62 elements, than $B = \{2, 3, \dots, 63\}$, but this set is not good, because sum of elements from B is 2015 and $5 \mid 2015$. B has less or equal than 61 elements. We are looking for the set, whose elements does not divide sum of them, so the best way to do that is making a sum of elements be a prime number. Sum $2 + 3 + \cdots + 63$ is equal 2015 and we have to remove at least one number, so we see that we can remove 4 and sum of remaining numbers will be 2011, which is prime number. Now, we conclude that set $B = \{2, 3, 5, 6, \dots, 63\}$ is good. The maximal number of elements of good subset of A is 61.

Problem 6. *[C6]* BPE

Let ABC be an equilateral triangle of side $k > 0$. On each side of the triangle we consider $n-1$ points, which divide the side to n equal line segments. We draw all the line segments ϵ_i with ends from the $n-1$ points we have considered on each side, not both on the same side, which are parallel to a side of the triangle. In this way the given equilateral triangle ABC is divided to n^2 equilateral triangles of side $\frac{k}{n}$. In the figure you can see the case with $n = 4$. We consider the set S which consists of the vertices of the triangle, the points we have considered on the three sides and the points of intersection of the line segments ϵ_i. With vertices from the set we consider rhombuses of two types as follows:

Rhombuses of type M are those of side length $\frac{k}{n}$ and rhombuses of type D are those of side length $\frac{2k}{n}$. Let m the number of rhombuses of type M and d the *number of rhombuses of type D. Find the difference $m - d$ with respect to n.*

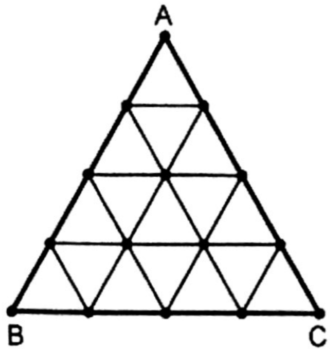


Figure 1

Solution

Each line segment with ends from the set S of length $\frac{k}{n}$ (not lying on a side of the triangle) is the diagonal one and only one rhombus of type M. The segments parallel to one side of the triangle are

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

Therefore the total number of rhombus of type M is:

$$m = 3\frac{n(n-1)}{2}$$

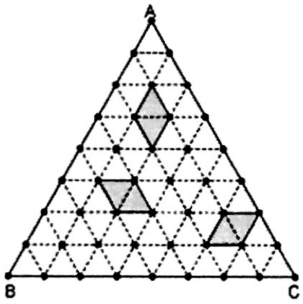


Figure 2

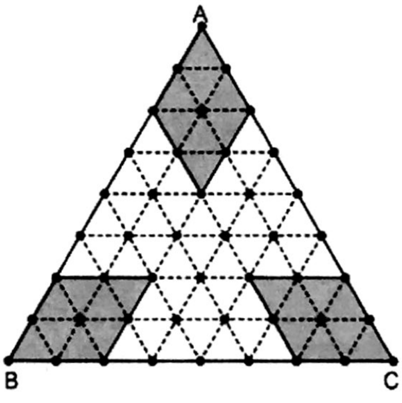


Figure 3

For the counting of rhombi of type D we distinguish the points of S which are interior points of the triangle ABC into three categories as follows:
The first category consists of points which are centers of one exactly rhombus of type D. These points are only 3, for every n. In figure (3) you can see the case for $n = 8$.

The second category consists of points which are centers of two exactly rhombi of type D. These points lie on the segments which are parallel to the sides of the triangle ABC and of the shortest possible distance from them. On each such segment there exist $n-4$ such points and therefore we have $3(n-4)$ points of this category. In figure (4) you can see these points for $n=8$.

The third category consists of the rest of points which are centers of three rhombi of type D. These points are totally:

$$1+2+3+\cdots+(n-5)=\frac{(n-5)(n-4)}{2}$$

In figure (5) you can see these points for $n=8$.

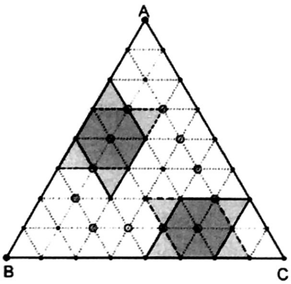


Figure 4

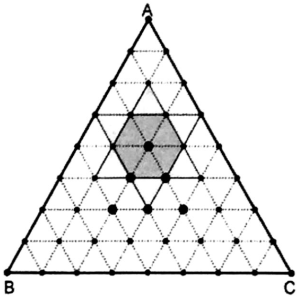


Figure 5

Hence the number of rhombi of type is the following:

$$d=3+3(n-4)2+3\frac{(n-5)(n-4)}{2}$$

$$\Leftrightarrow d=\frac{3}{2}\left(2+(n-1)(n-4)\right)$$

Finally we have $m-d=3(2n-3)$.

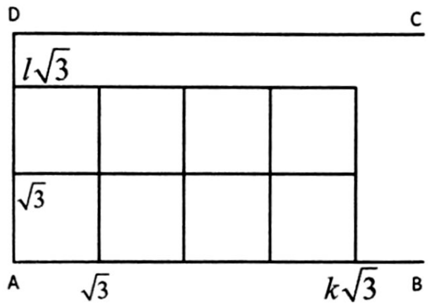
Problem 7. *[C7]* MNE

Consider a rectangle whose lengths of sides are natural numbers. If one places as many squares as possible, each with area 3, inside of a rectangle, so that their sides are parallel to the sides of a rectangle, then a maximal number of these squares fill exactly 50% of the area of a rectangle. Determine dimensions of all rectangles with this property.

Remark: $\sqrt{3}=1,73205\ldots$

Solution

Let $ABCD$ be a rectangle with $AB = M$ and $AD = n$ where m and n are natural numbers such that $m \geq n \geq 2$. Suppose that inside of a rectangle $ABCD$ is placed a rectangular lattice consisting of k identical squares whose areas are equals 3, where k of these squares are placed along the side AB, and l of these squares are placed along the side AD (see Figure below).



The sum of areas of all these squares is equal to $3kl$. Besides of the obvious conditions

$$k\sqrt{3} < m \qquad and \qquad l\sqrt{3} > n \tag{4}$$

by the assumption of the "maximality of lattice" consisting of these squares, we conclude that must be

$$(k+1)\sqrt{3} > m \qquad and \qquad (l+1)\sqrt{3} > n \tag{5}$$

Hence, the proposed problem is to determine all pairs m, n of natural numbers such that $m \geq n \geq 2$ for which the ratio

$$r_{m,n} := \frac{3kl}{mn} \tag{6}$$

is equal to $0,5$, where k and l are natural numbers exactly determined by the conditions (4) and (5).

Observe first that for $n \geq 6$, and so $m \geq n \geq 6$, using that by (5) hold $k\sqrt{3} > m - \sqrt{3}$ and $l\sqrt{3} > n - \sqrt{3}$, we have

$$r_{m,n} = \frac{k\sqrt{3} \cdot l\sqrt{3}}{mn} > \frac{(m-\sqrt{3})(n-\sqrt{3})}{mn}$$

$$= \left(1 - \frac{\sqrt{3}}{m}\right)\left(1 - \frac{\sqrt{3}}{n}\right) \geq \left(1 - \frac{\sqrt{3}}{n}\right)^2 \geq \left(1 - \frac{\sqrt{3}}{6}\right)^2 = 0.506... > 0.5$$

Hence, the required condition $r_{m,n} = 0.5$ *under the assumption* $m \geq n$ *yields* $n \leq 5$, *that is,* $n \in \{2,3,4,5\}$. *Now we consider these four possible cases.*

Case 1: $n = 2$ *Then* $l = 1$ *and thus as above we get*

$$r_{m,2} = \frac{3k}{2m} = \frac{\sqrt{3}\cdot k\sqrt{3}}{2m} > \frac{\sqrt{3}\cdot(m-\sqrt{3})}{2m} = \frac{\sqrt{3}}{2}\left(1-\frac{\sqrt{3}}{m}\right),$$

which is greater than 0,5 for each $m > (3\sqrt{3}+3)/2 = 4.098.....$ *Hence, must be* $m \in \{2,3,4\}$. *Immediate calculations give* $r_{2,2} = r_{2,4} = 3/4$ *(with* $k = 1,2$ *respectively) and* $r_{2,3} = 1/2$ *(with* $k = 1$*).*

Case 2: $n = 3$ *Then* $l = 1$ *and thus as above we get*

$$r_{m,3} = \frac{3k}{3m} = \frac{\sqrt{3}\cdot k\sqrt{3}}{3m} > \frac{\sqrt{3}\cdot(m-\sqrt{3})}{3m} = \frac{\sqrt{3}}{3}\left(1-\frac{\sqrt{3}}{m}\right),$$

which is greater than 0,5 for each $m > 4\sqrt{3}+6 = 12.928.....$ *Hence, must be* $m \in \{2,3,4,...,11,12\}$. *Direct calculations give* $r_{3,3} = 1/3$ *(with* $k = 2$*),* $r_{3,4} = r_{3,6} = r_{3,8} = r_{3,10} = r_{3,12} = 1/2$ *(with* $k = 2,3,4,5,6$, *respectively),* $r_{3,5} = 2/5$ *(with* $k = 2$*),* $r_{3,7} = 4/7$ *(with* $k = 4$ *),* $r_{3,9} = 5/9$ *(with* $k = 5$*) and* $r_{3,11} = 6/11$ *(with* $k = 6$*).*

Case 3: $n = 4$ *Then* $l = 2$ *and thus as above we get*

$$r_{m,4} = \frac{6k}{4m} = \frac{\sqrt{3}\cdot k\sqrt{3}}{2m} > \frac{\sqrt{3}\cdot(m-\sqrt{3})}{2m} = \frac{\sqrt{3}}{2}\left(1-\frac{\sqrt{3}}{m}\right),$$

which is greater than 0,5 for each $m > (3\sqrt{3}+3)/2 = 4.098....$. *Hence, must be* $m = 4$. *A calculation yields* $r_{4,4} = 3/4$ *(with* $k = 2$*).*

Case 4: $n = 5$ *Then* $l = 2$ *and thus as above we get*

$$r_{m,5} = \frac{6k}{5m} = \frac{2\sqrt{3}\cdot k\sqrt{3}}{5m} > \frac{2\sqrt{3}\cdot(m-\sqrt{3})}{5m} = \frac{2\sqrt{3}}{5}\left(1-\frac{\sqrt{3}}{m}\right),$$

which is greater than 0,5 for each $m > 12(4\sqrt{3}+5)/23 = 6.223....$ *Hence, must be* $m \in \{5,6\}$. *Calculation imply* $r_{5,5} = 12/25$ *(with* $K = 2$*) and* $r_{5,6} = 3/5$ *(with* $K = 3$*).*

Finally, from all the above cases we see that $r_{i,j} = 0,5$ *for* $(i,j) \in \{(2,3),(3,4),(3,6),(3,8),(3,10),(3,12)\}$. *These pairs present the dimensions of all rectangles with desired property.*

Problem 8. *[C8]* ROM

Determine the polygons with n *sides,* $n \geq 4$, *not necessarily convex, that satisfy* *the property that the reflection of every vertex of polygon with respect to every diagonal of the polygon does not fall outside the polygon.*

Note:*A diagonal is any segment joining two non-neighbouring vertices of the polygon; the reflection is considered with respect to the support line of the diagonal.*

Solution

A polygon with this property has to be convex, otherwise, we consider an edge of the convex hull of the set of the vertices that is not an edge of the polygon. All the other vertices are situated in one of the half-planes determined by the support-line of this edge, therefore the reflection of the other vertices falls outside the polygonal.

Now we fix a diagonal. It divides the polygon into two parts $p1$, $p2$. The reflection of $p1$ falls into the interior of $p2$ and vice versa. As a consequence, the diagonal is a symmetry axis for the polygon. Then every diagonal of the polygon bisects the angles of the polygonal and this means that there are 4 vertices and the polygon is a rhombus.

Any rhombus satisfies the desired condition.

Problem 9. *[C9]* RoM

Decide if it is possible to consider 2011 points in a plane such that:

i) the distance between every two of these points is different from 1, and

ii) every unit circle centered at one of these points leaves exactly 1005 of the points outside the circle.

Solution

It is not possible. If such a configuration would exist, the number of segments starting from any of the 2011 points towards the other ones and having length less than 1 would be 1005. Their total number would be $1005 \cdot 2011$. But each segment is counted twice, while $1005 \cdot 2011$ is odd, false.