Problema săptămânii 206

Arătaţi că dacă $a, b, c > 0$ verifică $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, atunci

$$\sqrt{\frac{1}{a^3+1}} + \sqrt{\frac{1}{b^3+1}} + \sqrt{\frac{1}{c^3+1}} \leq \frac{3}{\sqrt{2}}.$$

Olimpiada Tuymaada, 2014

Soluţia 1:

Avem $a^3 + 1 \geq a^2 + a$, $\forall\, a > 0$ (revine la $(a+1)(a-1)^2 \geq 0$, cu egalitate numai pentru $a = 1$) şi $\frac{1}{a+1} \leq \frac{1}{4}\left(1 + \frac{1}{a}\right)$, deci

$$\sqrt{\frac{1}{a^3+1}} \leq \frac{1}{\sqrt{a^2+a}} = \frac{\sqrt{2}}{\sqrt{2a(a+1)}} \leq \frac{\sqrt{2}}{2}\left(\frac{1}{2a} + \frac{1}{a+1}\right) \leq \frac{\sqrt{2}}{2}\left(\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{a}\right).$$

Adunând cu analoagele obţinem $\sqrt{\frac{1}{a^3+1}} + \sqrt{\frac{1}{b^3+1}} + \sqrt{\frac{1}{c^3+1}} \leq \frac{3}{\sqrt{2}}$, cu egalitate dacă şi numai dacă $a = b = c = 1$.

Soluţia 2: (*Ana Duguleanu*)

Folosind inegalitatea Cauchy-Buniakowsky-Schwarz, avem

$$\sqrt{\frac{1}{a^3+1}} + \sqrt{\frac{1}{b^3+1}} + \sqrt{\frac{1}{c^3+1}} \leq \sqrt{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{a}{a^3+1} + \frac{b}{b^3+1} + \frac{c}{c^3+1}\right)} =$$
$$\sqrt{3\left(\frac{a}{a^3+1} + \frac{b}{b^3+1} + \frac{c}{c^3+1}\right)} \leq \sqrt{3\left(\frac{a}{2a\sqrt{a}} + \frac{b}{2b\sqrt{b}} + \frac{c}{2c\sqrt{c}}\right)}.$$

Din inegalitatea dintre media aritmetică şi cea pătratică,

$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} \leq \sqrt{3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} = 3$, de unde rezultă inegalitatea din enunţ.

Soluţia 3: (user rightways pe AoPS)

Avem $\frac{1}{\sqrt{a^3+1}} \leq \frac{\sqrt{a+1}}{a^2+1} \leq \frac{\sqrt{a+1}}{2a} = \frac{2\sqrt{(a+1)2}}{4\sqrt{2}a} \leq \frac{(a+1)+2}{4\sqrt{2}a} = \frac{1}{4\sqrt{2}} + \frac{3}{4\sqrt{2}a}$

care, prin adunare cu analoagele, implică inegalitatea dorită.

Am primit soluţii de la: *Marin Hristov* (mixing variables), *David Ghibu, Albert Romaniuc, Carol Luca Gasan* şi *Radu Şerban.*

Problem of the week no. 206

If $a, b, c > 0$ satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, prove that

$$\sqrt{\frac{1}{a^3+1}} + \sqrt{\frac{1}{b^3+1}} + \sqrt{\frac{1}{c^3+1}} \le \frac{3}{\sqrt{2}}.$$

Tuymaada Olympiad, 2014

Solution 1:

We have $a^3 + 1 \ge a^2 + a \ \forall\, a > 0$ (it reduces to $(a+1)(a-1)^2 \ge 0$, with equality only for $a = 1$) and $\frac{1}{a+1} \le \frac{1}{4}\left(1 + \frac{1}{a}\right)$, hence

$$\sqrt{\frac{1}{a^3+1}} \le \frac{1}{\sqrt{a^2+a}} = \frac{\sqrt{2}}{\sqrt{2a(a+1)}} \le \frac{\sqrt{2}}{2}\left(\frac{1}{2a} + \frac{1}{a+1}\right) \le \frac{\sqrt{2}}{2}\left(\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{a}\right).$$

Adding this with its analogues yields $\sqrt{\frac{1}{a^3+1}} + \sqrt{\frac{1}{b^3+1}} + \sqrt{\frac{1}{c^3+1}} \le \frac{3}{\sqrt{2}}$, with equality if and only if $a = b = c = 1$.

Solution 2: (*Ana Duguleanu*)

Using Cauchy-Buniakowsky-Schwarz, we have

$$\sqrt{\frac{1}{a^3+1}} + \sqrt{\frac{1}{b^3+1}} + \sqrt{\frac{1}{c^3+1}} \le \sqrt{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{a}{a^3+1} + \frac{b}{b^3+1} + \frac{c}{c^3+1}\right)} =$$

$$\sqrt{3\left(\frac{a}{a^3+1} + \frac{b}{b^3+1} + \frac{c}{c^3+1}\right)} \le \sqrt{3\left(\frac{a}{2a\sqrt{a}} + \frac{b}{2b\sqrt{b}} + \frac{c}{2c\sqrt{c}}\right)}.$$

From the inequality between the arithmetic and quadratic means we get

$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} \le \sqrt{3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} = 3$, which leads to the desired inequality.

Solution 3: (user rightways on AoPS)

$$\frac{1}{\sqrt{a^3+1}} \le \frac{\sqrt{a+1}}{a^2+1} \le \frac{\sqrt{a+1}}{2a} = \frac{2\sqrt{(a+1)2}}{4\sqrt{2}a} \le \frac{(a+1)+2}{4\sqrt{2}a} = \frac{1}{4\sqrt{2}} + \frac{3}{4\sqrt{2}a}$$

added to its analogues, leads to the inequality to be proven.