Problema săptămânii 190
Fie $n \geq 3$ un număr natural şi $a_1, a_2, \ldots, a_n$ numere reale cu proprietatea că

$$a_k + a_{k+2} \geq 2a_{k+1}, \text{ oricare ar fi } k \in \{1, 2, \ldots, n-2\}.$$

Pentru orice $k \in \{1, 2, \ldots, n\}$ notăm cu $m_k = \dfrac{a_1 + a_2 + \ldots + a_k}{k}$.
Demonstraţi că $m_k + m_{k+2} \geq 2m_{k+1}$ oricare ar fi $k \in \{1, 2, \ldots, n-2\}$.

Soluţia 1:
Folosind definiţia lui m_j, inegalitatea de demonstrat se scrie

$$(a_1+a_2+\ldots+a_k)\left(\frac{1}{k}+\frac{1}{k+2}-\frac{2}{k+1}\right)+a_{k+1}\left(\frac{1}{k+2}-\frac{2}{k+1}\right)+a_{k+2}\cdot\frac{1}{k+2}\geq 0,$$

sau, eliminând numitorii,

$$2(a_1 + a_2 + \ldots + a_k) - (k^2 + 3k)a_{k+1} + (k^2 + k)a_{k+2} \geq 0.$$

Ultima inegalitate se scrie echivalent

$$\sum_{j=1}^{k} j(j+1)(a_j + a_{j+2} - 2a_{j+1}) \geq 0,$$

inegalitate evidentă.
Egalitatea are loc dacă şi numai dacă avem egalitate în fiecare din relaţiile $a_j + a_{j+2} - 2a_{j+1} \geq 0$, adică atunci când numerele $a_1, a_2, \ldots, a_n$ sunt în progresie aritmetică (adică sunt termeni consecutivi ai unei progresii aritmetice).

Remarcă: Nu este evidentă ultima rescriere a inegalităţii de demonstrat. Ea poate fi ghicită examinând cazurile ,,mici". O altă cale de a o vedea este următoarea: notând cu $r_k = a_{k+1} - a_k$ pentru orice k, relaţia din ipoteză ne arată că

$$r_1 \leq r_2 \leq \ldots \leq r_{n-1}.$$

Folosind că $a_k = a_1 + r_1 + r_2 + \ldots + r_{k-1}$, inegalitatea $2(a_1 + a_2 + \ldots + a_k) - (k^2 + 3k)a_{k+1} + (k^2 + k)a_{k+2} \geq 0$ revine la $k(k+1)r_{k+1} \geq 2(r_1 + r_2 + \ldots + r_k)$, adică la $2\sum_{j=1}^{k}(r_{k+1} - r_j) \geq 0$, inegalitate evidentă.

Soluţia 2: (*Carol Luca Gasan*)
Ne propunem să demonstrăm prin inducţie după $k \geq 2$ afirmaţia

$$P(k): \ m_{k-1} + m_{k+1} \geq 2m_k.$$

Pentru $k = 2$ avem de arătat că $a_1 + \dfrac{a_1 + a_2 + a_3}{3} \geq 2 \cdot \dfrac{a_1 + a_2}{2}$, adică $a_1 + a_3 \geq 2a_2$, relaţie adevărată.

Presupunând $P(k)$ adevărată, să demonstrăm $P(k+1)$.
Scriind că $a_j = jm_j - (j-1)m_{j-1}$, $\forall\, j$, avem că
$a_k + a_{k+2} \geq 2a_{k+1} \Leftrightarrow [km_k - (k-1)m_{k-1}] + [(k+2)m_{k+2} - (k+1)m_{k+1}] \geq 2[(k+1)m_{k+1} - km_k] \Rightarrow 3km_k + (k+2)m_{k+2} \geq (3k+3)m_{k+1} + (k-1)m_{k-1} \overset{P(k)}{\geq} (2k+4)m_{m+1} + 2(k-1)m_k$.
De aici, reducând termenii asemenea, obţinem $P(k+1)$.

Am mai primit soluţii de la: *David Andrei Anghel, Gabriel Turbincă* şi *Ana Valeria Duguleanu.*

Problem of the week no. 190
Let $n \geq 3$ be an integer and let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_k + a_{k+2} \geq 2a_{k+1}, \text{ for all } k \in \{1, 2, \ldots, n-2\}.$$

For all $k \in \{1, 2, \ldots, n\}$ we denote $m_k = \dfrac{a_1 + a_2 + \ldots + a_k}{k}$.
Prove that $m_k + m_{k+2} \geq 2m_{k+1}$ for all $k \in \{1, 2, \ldots, n-2\}$.

Solution 1:
Using the definition of m_j, we can write the inequality as

$$(a_1+a_2+\ldots+a_k)\left(\frac{1}{k} + \frac{1}{k+2} - \frac{2}{k+1}\right) + a_{k+1}\left(\frac{1}{k+2} - \frac{2}{k+1}\right) + a_{k+2} \cdot \frac{1}{k+2} \geq 0,$$

or, equivalently,

$$2(a_1 + a_2 + \ldots + a_k) - (k^2 + 3k)a_{k+1} + (k^2 + k)a_{k+2} \geq 0.$$

The last inequality can be written as

$$\sum_{j=1}^{k} j(j+1)(a_j + a_{j+2} - 2a_{j+1}) \geq 0,$$

which is obvious.
Equality holds if and only if each of the following inequalities are satisfied with equality
$a_j + a_{j+2} - 2a_{j+1} \geq 0$, which means that $a_1, a_2, \ldots, a_n$ are consecutive terms of an arithmetic progression.

Remark: It is not so easy to guess the last way to express the inequality. One can find it either by examining the small cases, or by introducing $r_k = a_{k+1} - a_k$ for all k. In terms of these numbers, the hypothesis translates to $r_1 \leq r_2 \leq \ldots \leq r_{n-1}$.
Using $a_k = a_1 + r_1 + r_2 + \ldots + r_{k-1}$, the inequality $2(a_1 + a_2 + \ldots + a_k) - (k^2 +$

$3k)a_{k+1} + (k^2 + k)a_{k+2} \geq 0$ reduces to $k(k+1)r_{k+1} \geq 2(r_1 + r_2 + \ldots + r_k)$, i.e. to $2\sum_{j=1}^{k}(r_{k+1} - r_j) \geq 0$, which is clear.

Solution 2: (*Carol Luca Gasan*)
We prove by induction after $k \geq 2$ the statement

$$P(k): \ m_{k-1} + m_{k+1} \geq 2m_k.$$

For $k = 2$ we must prove $a_1 + \dfrac{a_1 + a_2 + a_3}{3} \geq 2 \cdot \dfrac{a_1 + a_2}{2}$, i.e. $a_1 + a_3 \geq 2a_2$, which is true.
Assuming $P(k)$ to be true, let us prove $P(k+1)$.
Expressing $a_j = jm_j - (j-1)m_{j-1}$, $\forall\, j$, we have
$a_k + a_{k+2} \geq 2a_{k+1} \Leftrightarrow [km_k - (k-1)m_{k-1}] + [(k+2)m_{k+2} - (k+1)m_{k+1}] \geq 2[(k+1)m_{k+1} - km_k] \Rightarrow 3km_k + (k+2)m_{k+2} \geq (3k+3)m_{k+1} + (k-1)m_{k-1} \overset{P(k)}{\geq} (2k+4)m_{m+1} + 2(k-1)m_k$.
From here we obtain immediately $P(k+1)$.