THE THIRD JTST FOR JBMO - Saudi Arabia, 2017

Problem 1. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove the inequality

$$\frac{a(a-b^2)}{a+b^2} + \frac{b(b-c^2)}{b+c^2} + \frac{c(c-a^2)}{c+a^2} \geq 0.$$

Problem 2. Find all pairs of positive integers (p, q) such that both the equations $x^2 - px + q = 0$ and $x^2 - qx + p = 0$ have integral solutions.

Problem 3. Let BC be a chord of a circle (O) such that BC is not a diameter. Let AE be the diameter perpendicular to BC such that A belongs to the larger arc BC of (O). Let D be a point on the larger arc BC of (O) which is different from A. Suppose that AD intersects BC at S and DE intersects BC at T. Let F be the midpoint of ST and I be the second intersection point of the circle (ODF) with the line BC.
1. Let the line passing through I and parallel to OD intersect AD and DE at M and N, respectively. Find the maximum value of the area of the triangle MDN when D moves on the larger arc BC of (O) (such that $D \neq A$).
2. Prove that the perpendicular from D to ST passes through the midpoint of MN.

Problem 4. Consider a set S of 200 points on the plane such that 100 points are the vertices of a convex polygon A and the other 100 points are in the interior of the polygon. Moreover, no three of the given points are collinear. A triangulation is a way to partition the interior of the polygon A into triangles by drawing the edges between some two points of S such that any two edges do not intersect in the interior, and each point in S is the vertex of at least one triangle.
1. Prove that the number of edges does not depend on the triangulation.
2. Show that for any triangulation, one can color each triangle by one of three given colors such that any two adjacent triangles have different colors.

Solutions:

Problem 1. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove the inequality

$$\frac{a(a-b^2)}{a+b^2} + \frac{b(b-c^2)}{b+c^2} + \frac{c(c-a^2)}{c+a^2} \geq 0.$$

Solution:

We have $\dfrac{a(a-b^2)}{a+b^2} = \dfrac{a(a+b^2)-2ab^2}{a+b^2} = a - \dfrac{2ab^2}{a+b^2} \leq a - \dfrac{2ab^2}{2\sqrt{a}\,b} = a - b\sqrt{a}$.

Similarly, $\dfrac{b(b-c^2)}{b+c^2} \leq b - c\sqrt{b}$ and $\dfrac{c(c-a^2)}{c+a^2} \leq c - a\sqrt{c}$. Thus, it is sufficient to prove that

$$a+b+c \geq b\sqrt{a} + c\sqrt{b} + a\sqrt{c}.$$

By Cauchy-Schwarz, we have

$$(ab+bc+ca)(a+b+c) \geq \left(b\sqrt{a} + c\sqrt{b} + a\sqrt{c}\right)^2.$$

Therefore, it is enough to prove that

$$a+b+c \geq ab+bc+ca.$$

This inequality follows from

$$(a+b+c)^2 \geq 3(ab+bc+ca) \geq (a+b+c)(ab+bc+ca).$$

Equality holds if and only if $a=b=c=1$.

Problem 2. Find all pairs of positive integers (p,q) such that both the equations $x^2 - px + q = 0$ and $x^2 - qx + p = 0$ have integral solutions.

Solution:

Let $a, b \in \mathbb{Z}$ the solutions of the equation $x^2 - px + q = 0$. Then $a + b = p$ and $ab = q$, hence $a, b > 0$. Therefore $(a-1)(b-1) \geq 0$, i.e. $p - q + 1 \geq 0$.

Similarly, from the other equation, it follows that $q - p + 1 \geq 0$, hence $p - q \in \{-1, 0, 1\}$.

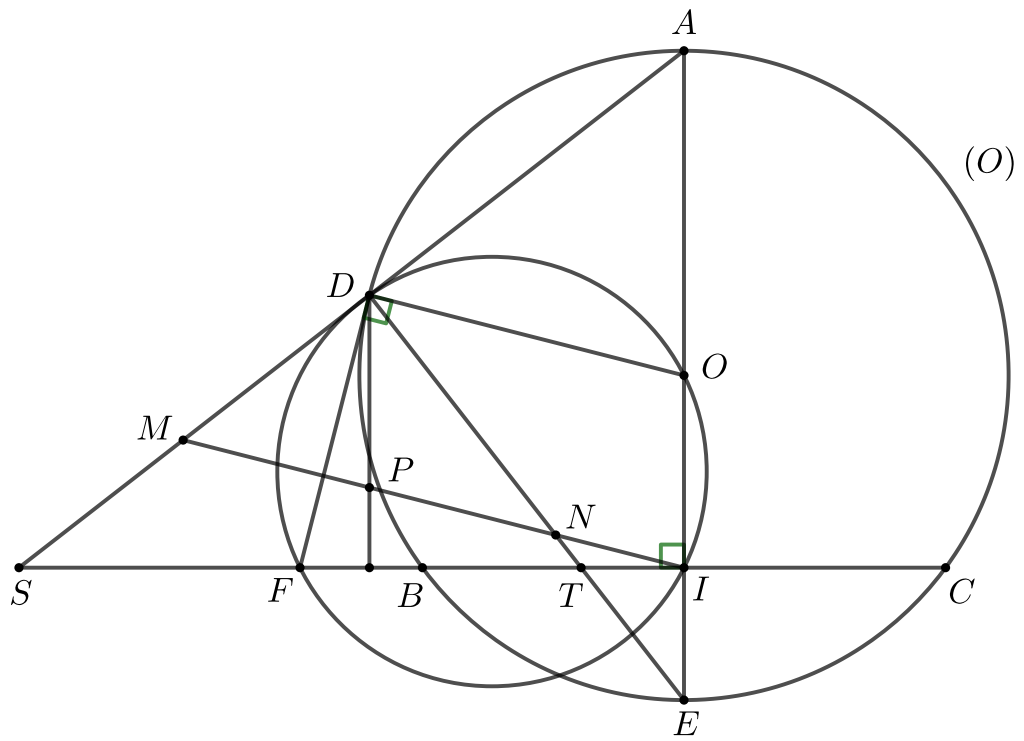
• If $p = q$, the equation $x^2 - px + p = 0$ has to have integer solutions. This means that $\Delta = p^2 - 4p$ has to be a perfect square. But $p^2 - 4p = k^2 \Leftrightarrow (p-2)^2 - k^2 = 4 \Leftrightarrow (p-2-k)(p-2+k) = 4$. Note that $p-2-k$ and $p-2+k$ have the same parity, so the only possibility is $p-2-k = p-2+k = 2$, i.e. $p = 4$. Indeed, for $p = 4$, the solutions of $x^2 - px + p = 0$ are both equal to 2, hence integers.

• If $p = q + 1$, the equation $x^2 - px + q = 0$ has the integer solutions 1 and q. The equation $x^2 - qx + p = 0$, which becomes $x^2 - qx + q + 1 = 0$, needs to have integer solutions. As above, $\Delta = q^2 - 4q - 4 = k^2 \Leftrightarrow (q-2-k)(q-2+k) = 8 \Leftrightarrow q - 2 - k = 2,\ q - 2 + k = 4 \Rightarrow q = 5$. We obtain $(p,q) = (6,5)$ which satisfies indeed the condition.

• Similarly, if $p - q = -1$ we obtain the pair $(p,q) = (5,6)$.

In conclusion, there are three convenient pairs: $(4, 4)$, $(5, 6)$ and $(6, 5)$.

Problem 3. Let BC be a chord of a circle (O) such that BC is not a diameter. Let AE be the diameter perpendicular to BC such that A belongs to the larger arc BC of (O). Let D be a point on the larger arc BC of (O) which is different from A. Suppose that AD intersects BC at S and DE intersects BC at T. Let F be the midpoint of ST and I be the second intersection point of the circle (ODF) with the line BC.
1. Let the line passing through I and parallel to OD intersect AD and DE at M and N, respectively. Find the maximum value of the area of the triangle MDN when D moves on the larger arc BC of (O) (such that $D \neq A$).
2. Prove that the perpendicular from D to ST passes through the midpoint of MN.
Solution:



1. First, note that $\angle ADE = 90°$, hence DO and DF are medians in the triangles ADE and SDT, respectively.
Then $\angle ODF = \angle ODT + \angle FDT = \angle OET + \angle FTD = 90°$. Hence, $\angle OIF = 180° - \angle ODF = 90°$, which implies that I is the midpoint of BC. Since ODE is isosceles triangle, it follows that INE and IMA are also isosceles, which implies that $IE = IN$ and $IM = IA$. Hence $MN = IM - IN = IA - IE = const.$
As triangles DMN and DAE are similar with a constant scale factor, it follows that, in order to maximize the area of DMN, we have to maximize the area of

triangle ADE. We have

$$[ADE] = \frac{1}{2} DA \cdot DE \leq \frac{DA^2 + DE^2}{4} = \frac{AE^2}{4}.$$

The equality occurs when $DA = DE$, i.e. D lies on the circle such that ADE is isosceles right triangle.
2. If P is the midpoint of MN, then

$$\angle PDN = \angle PND = \angle INE = \angle IEN,$$

thus $DP \parallel AE$, or the perpendicular line from D to ST passes through the midpoint of MN.

Problem 4. Consider a set S of 200 points on the plane such that 100 points are the vertices of a convex polygon A and the other 100 points are in the interior of the polygon. Moreover, no three of the given points are collinear. A triangulation is a way to partition the interior of the polygon A into triangles by drawing the edges between some two points of S such that any two edges do not intersect in the interior, and each point in S is the vertex of at least one triangle.
1. Prove that the number of edges does not depend on the triangulation.
2. Show that for any triangulation, one can color each triangle by one of three given colors such that any two adjacent triangles have different colors.
Solution:
1. Suppose that we have k triangles in some triangulation. By calculating the sum of all angles of these triangles, we have $180° \cdot k$.
The sum of interior angles of A is $180° \cdot 98$.
The sum of the angles around each of the 100 points situated in the interior of A is $360° \cdot 100$.
Hence, we have

$$180° \cdot k = 180° \cdot 98 + 360° \cdot 100 \Leftrightarrow k = 298.$$

Each triangle gives 3 edges and among them, there are 100 edges of A. Note that the interior edges are counted twice, hence the number of edges in each triangulation is

$$\frac{3 \cdot 298 - 100}{2} + 100 = 497.$$

2. We prove that for any polygon with n vertices containing m points in its interior such that no three of these $m+n$ points are collinear, and for any triangulation, we can color the triangles with 3 colors such that any two triangles sharing a common side have different colors.
We prove the assertion by induction after the number $m + n$ of total points.

If $m + n = 3$, then $n = 3$ and $m = 0$ and the conclusion is obvious.
Supposing the statement to be true for any configuration with less than $n + m$

points, consider a polygon $A_1A_2\dots A_n$ having m points in its interior and a triangulation.

• Suppose there is a vertex A_k such that $A_1A_2A_k$ is one of the triangles of the triangulation. If $k = 3$, the polygon $A_1A_3A_4\dots A_n$ is triangulated and can, by the induction hypothesis for $n + m - 1$ total points, be colored conveniently. For the triangle $A_1A_2A_3$ one can choose a color, different from the color already given to the triangle containing $[A_1A_3]$.

We proceed similarly for the case when $A_k = A_n$. In the remaining cases, consider the polygons $A_2A_3\dots A_k$ and $A_kA_{k+1}\dots A_1$. Both are triangulated and, according to the induction hypothesis, they can be colored conveniently. We can still find a suitable color for the triangle $A_1A_2A_k$, different from the colors of the triangles containing $[A_1A_k]$ and $[A_2A_k]$.

• There must be a point X such that the triangle A_1A_2X is one of the triangles of the triangulation. If X is not a vertex, it must be an interior point. In the triangulation, X must be joined by segments with at least 3 points (among those being A_1 and A_2; the sum of the angles around X must be $360°$). Consider the polygon $P = A_1A_2B_1B_2\dots B_j$ determined by the points joined with X. Then X is the only point of those $n + m$ interior to this polygon. Now remove the point X. The triangulation in the interior of P being thus destroyed, we consider an arbitary triangulation of P and keep the rest of the initial triangulation. We obtain a triangulation for an n-gon having $m-1$ interior points. According to the inductive hypothesis, we can color conveniently this triangulation. Now we put X back and get back to our initial configuration. We keep the coloring of the triangles outside P (if any), and repaint the interior of P according to our triangulation. We have to color the triangles XA_2B_1, XB_1B_2, … , XB_jA_1 and XA_1A_2. For XA_2B_1 there is at most one forbidden color, the one of the already colored triangle containing the side $[A_2B_1]$ (if such a triangle exists). For XB_1B_2 there are at most two forbidden colors, the ones of the triangle XA_2B_1 and the color of the triangle exterior to P that contains the side $[B_1B_2]$. For each of the following triangles, there are at most two forbidden colors, so there is always an available color to use. Finally, for the triangle XA_1A_2, there is an extra restriction: its color needs to be different from the color of not only XB_jA_1, but also from the color of XA_2B_1. Fortunately, there is no extra restriction from the exterior of $[A_1A_2]$, so there is still a third available color.

This concludes our induction.