THE SECOND JTST FOR JBMO - Saudi Arabia, 2017

Problem 1. Given a polynomial $f(x) = x^4+ax^3+bx^2+cx$. It is known that each of the equations $f(x) = 1$ and $f(x) = 2$ has four real roots (not necessarily distinct). Prove that if the roots of the first equation satisfy the equality $x_1 + x_2 = x_3 + x_4$, then the same equation holds for the roots of the second equation.

Problem 2. A positive integer $k > 1$ is called nice if for any pair (m, n) of positive integers satisfying the condition $kn + m \mid km + n$ we have $n \mid m$.
1. Prove that 5 is a nice number.
2. Find all the nice numbers.

Problem 3. Let (O) be a circle, and BC be a chord of (O) such that BC is not a diameter. Let A be a point on the larger arc BC of (O), and let E, F be the feet of the perpendiculars from B and C to AC and AB, respectively.
1. Prove that the tangents to (AEF) at E and F intersect at a fixed point M when A moves on the larger arc BC of (O).
2. Let T be the intersection of EF and BC, and let H be the orthocenter of ABC. Prove that TH is perpendicular to AM.

Problem 4. Find the number of ways one can put numbers 1 or 2 in each cell of an 8×8 chessboard in such a way that the sum of the numbers in each column and in each row is an odd number. (Two ways are considered different if the number in some cell in the first way is different from the number in the cell situated in the corresponding position in the second way).

Solutions:

Problem 1. Given a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx$. It is known that each of the equations $f(x) = 1$ and $f(x) = 2$ has four real roots (not necessarily distinct). Prove that if the roots of the first equation satisfy the equality $x_1 + x_2 = x_3 + x_4$, then the same equation holds for the roots of the second equation.

Solution:

Consider the equation $f(x) = 1$, i.e. $x^4 + ax^3 + bx^2 + cx = 1$. Since it has four roots, x_1, x_2, x_3, x_4, we can write it as $(x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0$. Note that $x_1 + x_2 = x_3 + x_4 = -\dfrac{a}{2}$. We can rewrite the equation

$$\left(x^2 - (x_1 + x_2)x + x_1x_2\right)\left(x^2 - (x_3 + x_4)x + x_3x_4\right) = 0$$

or

$$\left(x^2 + \frac{a}{2}\,x + x_1x_2\right)\left(x^2 + \frac{a}{2}\,x + x_3x_4\right) = 0.$$

The equation $f(x) = 2$ can be written as $\left(x^2 + \dfrac{a}{2}\,x + x_1x_2\right)\left(x^2 + \dfrac{a}{2}\,x + x_3x_4\right) = 1$. Putting $y = x^2 + \dfrac{a}{2}\,x$, we get $(y + x_1x_2)(y + x_3x_4) = 1$. Since $f(x) = 2$ has four real roots, this equation has two roots, say $y = \alpha$ and $y = \beta$. This implies that the equations $x^2 + \dfrac{a}{2}\,x = \alpha$ and $x^2 + \dfrac{a}{2}\,x = \beta$ have two solutions each. It follows that the four roots, x_1', x_2', x_3', x_4', of the equation $f(x) = 2$ can be divided into two pairs that have the sum equal to $-\dfrac{a}{2}$, which means $x_1' + x_2' = x_3' + x_4'$. This finishes the proof.

Problem 2. A positive integer $k > 1$ is called nice if for any pair (m, n) of positive integers satisfying the condition $kn + m \mid km + n$ we have $n \mid m$.

1. Prove that 5 is a nice number.
2. Find all the nice numbers.

Solution:

1. For $k = 5$, we need to prove that for all m, n satisfying $5n + m \mid 5m + n$ we have $n \mid m$. Note that $5n + m \le 5m + n$ means $n \le m$, hence $1 \le \dfrac{5m + n}{5n + m} < 5$. Then $A = \dfrac{5m + n}{5n + m} \in \{1, 2, 3, 4\}$. We consider the cases:

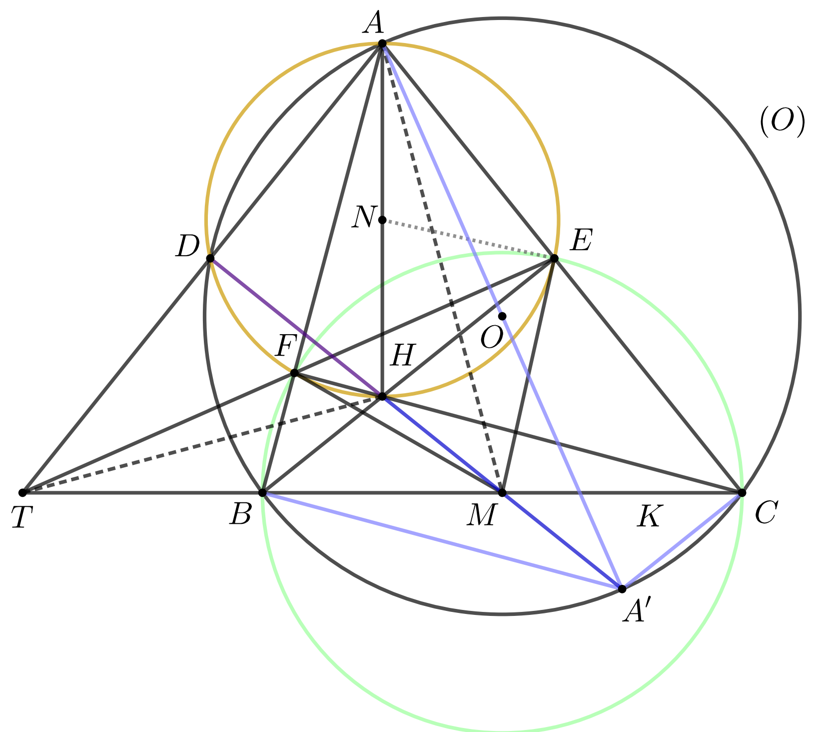
- If $A = 1$ then $m = n$.
- If $A = 2$ then $5m + n = 10n + 2m$, i.e. $m = 3n$.
- If $A = 3$ then $5m + n = 15n + 3m$ and $m = 7n$.
- If $A = 4$ then $5m + n = 20n + 4m$, i.e. $m = 19n$.

So in all cases, we always have $n \mid m$, which implies that $k = 5$ is a nice number.

2. We can directly check that $k = 2$ is a nice number. Consider some nice number $k > 2$. As above, it follows that $n \le m$ and $1 \le \dfrac{km + n}{kn + m} < k$. Thus $A = \dfrac{km + n}{kn + m} \in \{1, 2, 3, \ldots, k - 1\}$. In case $A = 2$, we have $km + n = 2m + 2kn$,

or $\frac{m}{n} = \frac{2k-1}{k-2}$. We must have $\frac{m}{n} \in \mathbb{Z}$, i.e. $\frac{2k-1}{k-2} = 2 + \frac{3}{k-2} \in \mathbb{Z}$. As $k > 2$, it follows that $k - 2 \in \{1, 3\}$, or $k \in \{3, 5\}$. It is easy to check that 3 is indeed a nice number. In conclusion, the only nice numbers are 2, 3 and 5.

Problem 3. Let (O) be a circle, and BC be a chord of (O) such that BC is not a diameter. Let A be a point on the larger arc BC of (O), and let E, F be the feet of the perpendiculars from B and C to AC and AB, respectively.
1. Prove that the tangents to (AEF) at E and F intersect at a fixed point M when A moves on the larger arc BC of (O).
2. Let T be the intersection of EF and BC, and let H be the orthocenter of ABC. Prove that TH is perpendicular to AM.



Solution:
1. Denote by M the midpoint of BC. Since $\angle AEH = \angle AFH = 90°$, points A, H, E and F belong to the same circle and the center of this circle is the midpoint N of AH. It is easy to check that $NE = NH$ and $ME = MB$, so $\angle MEN = \angle MEB + \angle NEB = \angle MBE + \angle NHE = \angle EAH + \angle AHE = 90°$.
Then ME is the tangent line of (AEF) at E. By same way, we have that FM is the tangent line of (AEF) at F. These imply that the tangent lines of (AEF) at E and F meet at the fixed point M.
2. Consider the diameter AA' of (O). It is easy to see that $BHCA'$ is a parallelogram and H, M and A' are collinear.
Suppose that $MH \cap (O) = D \neq A'$. Then $\angle ADH = \angle ADA' = 90°$, which means that $D \in (AEF)$.

By considering the three radical axis of three circles, (O), $(BFEC)$ and $(AEHF)$, we have that AD, EF and BC are concurrent at T, which is the radical center. In triangle ATM, we have $AH \perp TM$ and $MH \perp AT$, hence H is its orthocenter. From this, we can conclude that $TH \perp AM$.

Problem 4. Find the number of ways one can put numbers 1 or 2 in each cell of an 8×8 chessboard in such a way that the sum of the numbers in each column and in each row is an odd number. (Two ways are considered different if the number in some cell in the first way is different from the number in the cell situated in the corresponding position in the second way).

Solution:

Consider the leftmost column and lowest row of table; we color all these cells. We can see that for every way one can fill with numbers the sub-square 7×7 that is not colored, we can choose the numbers for the correspondent colored position at same column or row. Indeed, if the sum of the 7 numbers is odd, then we put 2 on the remaining cell; otherwise, we put 1. Finally, the number in the colored corner cell can be chosen based on the parity of the sum of all number in the square 7×7. These means that the way to fill in the square 7×7 uniquely defines the numbers in the remaining cells. Since we can fill each cell among the 49 cells of the 7×7 square by 1 or 2 in any way, the number of ways is 2^{49}.