THE FIRST JTST FOR JBMO - Saudi Arabia, 2017

Problem 1. Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality

$$\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \geq 1 + a + b + c.$$

Problem 2. Find all prime numbers p such that $\dfrac{3^{p-1}-1}{p}$ is a perfect square.

Problem 3. On the table, there are 1024 marbles and two students, A and B, alternatively take a positive number of marble(s). The student A goes first, B goes after that and so on. On the first move, A takes k marbles with $1 < k < 1024$. On the moves after that, A and B are not allowed to take more than k marbles or 0 marbles. The student that takes the last marble(s) from the table wins. Find all values of k the student A should choose to make sure that there is a strategy for him to win the game.

Problem 4. Let ABC be an acute, non isosceles triangle and (O) be its circumcircle (with center O). Denote by G the centroid of the triangle ABC, by H the foot of the altitude from A onto the side BC and by I the midpoint of AH. The line IG intersects BC at K.
1. Prove that $CK = BH$.
2. The ray $(GH$ intersects (O) at L. Denote by T the circumcenter of the triangle BHL. Prove that AO and BT intersect on the circle (O).

Solutions:

Problem 1. Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality

$$\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \geq 1 + a + b + c.$$

Solution:
First, we can see that among three numbers $a-1$, $b-1$, $c-1$, there are two numbers that have the same sign. Without loss of generality, we may assume that $a-1$ and $b-1$ have the same sign, which means that $(a-1)(b-1) \geq 0$. By applying Cauchy-Schwarz inequality, we have

$$\sqrt{2(1+c^2)} = \sqrt{(1^2+1^2)(1^2+c^2)} \geq 1 + c$$

and

$$\sqrt{(1^2+a^2)(1^2+b^2)} \geq 1 + ab.$$

Hence, it is enough to prove that $(1+c)(1+ab) \geq 1+a+b+c$ which is equivalent to $1+ab+c+abc \geq 1+a+b+c$ or $(a-1)(b-1) \geq 0$.

Problem 2. Find all prime numbers p such that $\dfrac{3^{p-1}-1}{p}$ is a perfect square.

Solution:
Let p be a prime satisfying the condition of the problem. By assumption, there is a positive integer A such that $3^{p-1} - 1 = pA^2$. It is clear that $p = 2$ is a solution. Now, we consider $p > 2$. Put $p - 1 = 2k$; one has $(3^k - 1)(3^k + 1) = pA^2$. Since $(3^k - 1,\ 3^k + 1) = 2$, it follows that there are positive integers B and C such that either

$$3^k - 1 = 2pB^2,\ 3^k + 1 = 2C^2$$

or

$$3^k - 1 = 2B^2,\ 3^k + 1 = 2pC^2.$$

But, the first case cannot hold since $2C^2 = 3^k + 1 \equiv 1 \pmod 3$ means $C^2 \equiv 2 \pmod 3$, which is impossible.
For the second case, if k is odd, then $4 \mid 3^k + 1 = 2pC^2$, hence $2 \mid C$ (since p is odd). This means that $3^k + 1 = 2pC^2$ is divisible by 8 which contradicts $3^k + 1 \equiv 4 \pmod 8$ (since k is odd). Thus, k must be even.
Put $k = 2m$; then $2B^2 = 3^k - 1 = (3^m - 1)(3^m + 1)$. Again, since $(3^m - 1,\ 3^m + 1) = 2$, there are positive integers D and E such that either

$$3^m - 1 = E^2,\ 3^m + 1 = 2D^2$$

or

$$3^m - 1 = 2E^2,\ 3^m + 1 = D^2.$$

As above, the equality $3^m + 1 = 2D^2$ leads to a contradiction. Hence, $3^m + 1 = D^2$, that is $3^m = D^2 - 1 = (D-1)(D+1)$. Therefore, there are non-negative integers $t > s$ such that $D - 1 = 3^s$ and $D + 1 = 3^t$. This gives $2 = (D+1) - (D-1) = 3^t - 3^s = 3^s(3^{t-s} - 1)$. This happens if and only if $3^s = 1$ and $3^{t-s} - 1 = 2$, i.e. $s = 0$ and $t = 1$, we find that $p = 5$ (which satisfies indeed the given condition). In conclusion, $p = 2$ and $p = 5$.

Problem 3. On the table, there are 1024 marbles and two students, A and B, alternatively take a positive number of marble(s). The student A goes first, B goes after that and so on. On the first move, A takes k marbles with $1 < k < 1024$. On the moves after that, A and B are not allowed to take more than k marbles or 0 marbles. The student that takes the last marble(s) from the table wins. Find all values of k the student A should choose to make sure that there is a strategy for him to win the game.

Solution:

After the first move, there are $1024 - k$ marbles remaining and neither student can take more than k marbles on their turn. We shall prove that if $k+1 \mid 1024-k$ then student A has a winning strategy. Indeed, let $1024 - k = m(k+1)$ with $m \in \mathbb{N}$, $m > 0$. On each turn, if B chooses x marble(s), then A will choose $k + 1 - x$ marble(s) on his next turn. Since $x \in \{1, 2, 3, \ldots, k\}$, then $k + 1 - x \in \{1, 2, 3, \ldots, k\}$ which implies that A has always a move available. After m turns of A and m turns of B, student A takes the last marble and wins the game.

If $1024 - k$ is not divisible by $k + 1$ then we put $1024 - k = m(k+1) + n$ with $0 < n < k + 1$. In this case, student B can take n marbles on his next turn. The number of remaining marbles is $m(k+1)$ then, by applying the same argument as above, student B has a strategy to win the game.

Hence, the condition we have to find is $k + 1 \mid 1024 - k$ or $k + 1 \mid 1025$. Note that the divisors of 1025 are 1, 5, 25, 41, 205, 1025, so $k + 1 \in \{1, 5, 25, 41, 205, 1025\}$ and $0 < k < 1024$. This means that the convenient values of k are 4, 24, 40, 204.

Problem 4. Let ABC be an acute, non isosceles triangle and (O) be its circumcircle (with center O). Denote by G the centroid of the triangle ABC, by H the foot of the altitude from A onto the side BC and by I the midpoint of AH. The line IG intersects BC at K.

1. Prove that $CK = BH$.

2. The ray $(GH$ intersects (O) at L. Denote by T the circumcenter of the triangle BHL. Prove that AO and BT intersect on the circle (O).

Solution:

1. Let M be the midpoint of BC; then $G \in AM$ and $\dfrac{AG}{AM} = \dfrac{2}{3}$. Take the point $K' \in BC$ such that M is the midpoint of HK'; then AM is the median of triangle AHK' and G is its centroid. Then $K'G$ is the median of triangle AHK' or $K'G$ passes through the midpoint of AH. This implies that $K \equiv K'$ and we have

$BH = CK$.
2. The line passing through A and parallel to BC intersects again the circle (O) at E. Then by the symmetry with respect to perpendicular bisector of BC, it is easy to check that $AHKE$ is a rectangle. Since $\dfrac{GA}{GM} = \dfrac{AE}{HM} = 2$, the points H, G, E and L are collinear.
Then $\angle BLH \equiv \angle BLE \equiv \angle BCE \equiv \angle ABC$ which implies that $\angle ABT = \angle ABC + \angle CBT = \angle BLH + \angle CBT = 90°$. Thus, if we denote $\{D\} = BT \cap (O)$, then AD is the diameter of (O), hence $O \in AD$. Therefore, BT and AO intersect at a point that belongs to (O).