Stars of Mathematics 2019, Juniors' Competition – Solutions

Problem 1. Determine the positive integers n that satisfy the following property: for every positive divisor d of n, $d+1$ is a divisor of $n+1$.

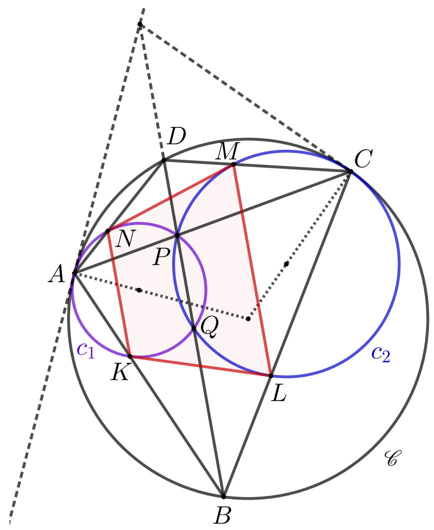
Solution:
We prove that the numbers that have the given property are 1 and the odd prime numbers. It is clear that all these numbers do indeed have the desired property and also that 2 does not have it.
Conversely, let us consider a composite number n and prove that it does not have the given property. If n is composite, then $n = ab$ cu $1 < a \leq b < n$. It follows that $b+1$ divides $n+1$, i.e. there exists $c \in \mathbb{Z}$ such that $c(b+1) = n+1 = ab+1$. We obtain that b divides $c-1$. Obviously, $c > 1$. We deduce that $c - 1 \geq b$, i.e. $c \geq b+1$. Then $ab+1 = c(b+1) \geq (b+1)^2$, which means that $ab+1 \geq b^2+2b+1$, leading to $a \geq b+2$, which contradicts $a \leq b$.
In conclusion, no composite number does satisfy the requirements.

Problem 2. Let A and C be two points on a circle $\mathscr{C}$ such that (AC) is not a diameter, and let P be a point of the line segment (AC), other then its midpoint. Circles c_1 and c_2 are interiorly tangent to the circle $\mathscr{C}$ at A and C, respectively. They both pass through P and intersect again at Q. The line PQ intersects circle $\mathscr{C}$ at B and D. Circle c_1 intersects line segments AB and AD at K and N, respectively, while circle c_2 intersects line segments CB and CD at L and M, respectively. Prove that:
a) the quadrilateral $KLMN$ is an isosceles trapezoid;
b) Q is the midpoint of the line segment BD.

Thanos Kalogerakis

Solution 1: a) Point B lies on the radical axis, BD, of circles c_1 and c_2, therefore $BK \cdot BA = BL \cdot BC$, which indicates that the quadrilateral $AKLC$ is cyclic. So is $ACMN$. It follows that $\angle LKB = \angle ACB$. The angle between the tangent at A to the circle c_1 (which is also tangent to $\mathscr{C}$) with line AB subtends arcs AK of circle c_1 and AB of circle $\mathscr{C}$, and therefore $\angle ANK = \angle ADB$. We obtain that $NK \parallel BD$ and, similarly, $ML \parallel BD$. (This fact also follows from the homotheties that transform circles c_1, and c_2, respectively, into $\mathscr{C}$.) Finally, $\angle NKL = 180^\circ - \angle AKN - \angle LKB = 180^\circ - \angle ABD - \angle LKB = 180^\circ - \angle ACD - \angle ACB = \angle BAD$. Similarly, $\angle MNK = \angle BAD$, which leads to the conclusion. Alternatively, one could have noticed that line segments ML, PQ and $[NK]$ share the same perpendicular bisector.
b) The radical axes of circles c_1, c_2, $\mathscr{C}$ (one for each pair of circles), i.e. the tangent line to $\mathscr{C}$ at A, the tangent line to $\mathscr{C}$ at C and the line BD are not all parallel, therefore they are concurrent in the radical center. Diagonal BD is then a symmedian of triangle ABC, which means that quadrilateral $ABCD$ is harmonic. (One

can also use this fact to give a different proof to a).)
As $NPQK$ is an isosceles trapezoid, it follows that $\angle NAP = \sphericalangle QAK$, which means that rays $(AP$ and $(AQ$ are isogonal with respect to angle $\angle DAB$. But $ABCD$ being harmonic, AP is a symmedian of triangle DAB, therefore AQ, which is its isogonal, is the median.



Another idea:
One could use a property of harmonic quadrilaterals that was put into evidence by the Danube Mathematical Competition from the same year:
If R is the midpoint of diagonal BD of a harmonic quadrilateral $ABCD$, then $\angle ARD = \angle CRD$.
One proves that R is the only point on the line segment BD that has the property from above. Next, one proves that point Q does have the property, which makes it the midpoint of BD.

Problem 3. On the board are written initially three consecutive positive integers, $n-1$, n, $n+1$. A move consists of choosing two numbers written on the board a and b, and replacing them with $2a-b$ and $2b-a$. For what values of n is it possible to obtain, after a succession of such moves, that two of the numbers written on the board are equal to 0?

Andrei Eckstein

Solution 1: (*Alexandru Mihalcu*)
We prove that we can obtain two 0-s on the board if and only if n is a power of 3. As the sum of the numbers written on the board stays the same, we must obtain on the board the numbers $(0, 0, 3n)$. If $p \neq 3$ is a prime divisor of n, then, in the final configuration, all the numbers are divisible by p. But if $p \mid 2a-b$, $p \mid 2b-a$ then $p \mid \big(2(2a-b)+(2b-a)\big)$, i.e. $p \mid 3a$. As $(p, 3) = 1$, it follows that $p \mid a$ and then $p \mid b$. From the above, we deduce that if in the end all the numbers are

divisible by a prime $p \neq 3$, then they were always divisible by that prime. Since g.c.d.$(n-1, n, n+1) = 1$, it follows that one cannot obtain two 0-s on the board if n has prime divisors other than 3.
We are left with the case when $n = 3^k$, cu $k \in \mathbb{N} \cup \{0\}$. For $k = 0$, starting from $(0, 1, 2)$, it is easy to get to $(0, 0, 3)$. Assume $k \geq 1$. Choosing at the first move $a = n-1$, $b = n+1$, we obtain on the board numbers $n-3, n, n+3$, all multiples of 3. They can be written as $3(m-1), 3m, 3(m+1)$ and the subsequent moves function as if the numbers written on the board were $m-1, m, m+1$. (After $j \leq k$ moves, on the board will be the numbers $3^k - 3^j$, 3^k, $3^k + 3^j$. After move no. k we get 0, 3^k, $2 \cdot 3^k$. Choosing $a = 3^k$ and $b = 2 \cdot 3^k$ we obtain on the board $0, 0, 3^{k+1}$.

Solution 2: (*Andrei Eckstein*)
We say that a triple (a, b, c) is solvable if, starting with the numbers a, b, c written on the board, one can obtain, after a succession of moves, two 0-s on the board.
Then (a, b, c) is solvable if and only if (ta, tb, tc), $t \in \mathbb{N}$, is solvable (one performs tha same moves, only the "unit of measurement" changes). (*)
We prove that the triple $(n-1, n, n+1)$ is solvable if and only if n is a power of 3.
Let us first prove that if $n = 3^m$, $m \in \mathbb{N} \cup \{0\}$, then the triple $(n-1, n, n+1)$ is solvable. We prove the statement by induction after $m \geq 0$.
For $m = 0$: he triple $(0, 1, 2)$ is solvable by a single move, choosing $a = 1$, $b = 2$.
Assuming the statement to be true for m, let us prove it for $m+1$. Having on the board the triple $(3^{m+1} - 1, 3^{m+1}, 3^{m+1} + 1)$, we choose $a = 3^{m+1} - 1$ and $b = 3^{m+1} + 1$ and we get to the triple $(3^{m+1} - 3, 3^{m+1}, 3^{m+1} + 3)$. According to the remark (*), this triple is solvable because the inductive hypothesis tells us that the triple $(3^m - 1, 3^m, 3^m + 1)$ is solvable.
Let us notice that the sum of the numbers written on the board does not change while performing a move. It remains $3n$, which is a multiple of 3. After the first move, all the numbers become equal modulo 3 because $2a - b \equiv -a - b \equiv 2b - a \pmod 3$. If, after the first move, they became all congruent to 1 or 2 mod 3, they will remain that way, so they can not become 0. Thus, it is mandatory that the first move makes all the numbers on the board multiples of 3. Also, if (a, b, c) is solvable, i.e. can be transformed into $(0, 0, a+b+c)$ through a succession of moves, then, before the last move, the numbers written on the board have to be 0, $\dfrac{a+b+c}{3}$ and $\dfrac{2(a+b+c)}{3}$, which shows that it is necessary to have $3 \mid a+b+c$.
If $n = 3k+1$, then the last move must be $(3k, 3k+1, 3k+2) \mapsto (3k, 3k, 3k+3)$. If $k = 0$, we are done ($n = 1$ is a power of 3); otherwise, this triple is solvable if and only if $(k, k, k+1)$ is solvable. But this triple is not solvable because the sum $k + k + (k+1)$ is not divisible by 3.
If $n = 3k+2$ then the first move must be $(3k+1, 3k+2, 3k+3) \mapsto (3k, 3k+3, 3k+3)$. This triple is solvable if and only if $(k, k+1, k+1)$ is solvable. But this triple is not solvable because the sum $k + (k+1) + (k+1)$ is not divisible by 3.
If $n = 3k$, the first move must be $(3k-1, 3k, 3k+1) \mapsto (3k-3, 3k, 3k+3)$. This

second position is solvable if and only if $(k-1, k, k+1)$ is solvable. As $n \neq 0$, there exist $u, v \in \mathbb{N}$ such that $n = 3^u \cdot v$, $(v, 3) = 1$. Repeating this reasoning, after u moves we get to the triple $(v-1, v, v+1)$. We have seen that the only triple of this form that is solvable is $(0, 1, 2)$, therefore it is necessary for n to be a power of 3.

Problem 4. Prove that, if positive real numbers $a_1, a_2, \ldots, a_n$ have the product 1, then

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} + \ldots + \left(\frac{a_{n-1}}{a_n}\right)^{n-1} + \left(\frac{a_n}{a_1}\right)^{n-1} \geq a_1^2 + a_2^2 + \ldots + a_n^2.$$

Andrei Eckstein

Solution: We prove that, for all $a_1, a_2, \ldots, a_n > 0$, the following inequality holds

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} + \ldots + \left(\frac{a_{n-1}}{a_n}\right)^{n-1} + \left(\frac{a_n}{a_1}\right)^{n-1} \geq \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{\sqrt[n]{a_1^2 a_2^2 \ldots a_n^2}}.$$

This inequality is obtained by adding the inequality below with its analogues obtained by cyclic permutation of the variables:

$$(n-1) \cdot \left(\frac{a_1}{a_2}\right)^{n-1} + (n-2) \cdot \left(\frac{a_2}{a_3}\right)^{n-1} + \cdots + \left(\frac{a_{n-1}}{a_n}\right)^{n-1} \overset{medii}{\geq}$$

$$\frac{n(n-1)}{2} \cdot \left(\left(\frac{a_1}{a_2}\right)^{n-1} \left(\frac{a_2}{a_3}\right)^{n-2} \cdot \ldots \cdot \left(\frac{a_{n-1}}{a_n}\right)\right)^{\frac{2}{n}} = \frac{n(n-1)}{2} \left(\frac{a_1^n}{a_1 a_2 a_3 \ldots a_n}\right)^{\frac{2}{n}}.$$

Equality holds if all the numbers are equal to 1.

Remarks: For $n = 3$ this inequality has been published by Šefket Arslanagić in Elemente der Mathematik; for $n = 4$ it appears, in a weaker form, in Mathscope, pb 321.1 (author Lê Thanh Hâi)).