Problema săptămânii 186

Arătaţi că dacă $x_1, x_2, \ldots, x_9 > 0$, atunci are loc inegalitatea

$$\frac{x_1 - x_3}{x_1x_3+2x_2x_3+x_2^2}+\frac{x_2 - x_4}{x_2x_4+2x_3x_4+x_3^2}+\ldots+\frac{x_8 - x_1}{x_8x_1+2x_9x_1+x_9^2}+\frac{x_9 - x_2}{x_9x_2+2x_1x_2+x_1^2}\geq 0.$$

Olimpiadă Rusia, 2019-2020

Soluţia 1: (*Orestes Lignos*, preluată de pe https://mathematica.gr)

Avem $\dfrac{x_1 - x_3}{x_1x_3 + 2x_2x_3 + x_2^2} = \dfrac{1}{x_3} - \dfrac{(x_2 + x_3)^2}{x_3(x_1x_3 + 2x_2x_3 + x_2^2)}$ şi analoagele.

Inegalitatea se scrie $\displaystyle\sum \frac{(x_2 + x_3)^2}{x_3(x_1x_3 + 2x_2x_3 + x_2^2)} \leq \sum \frac{1}{x_1}$ (1).

Din Cauchy-Schwarz, $\dfrac{x_2}{x_2 + x_3} + \dfrac{x_3}{x_1 + x_2} = \dfrac{x_2^2}{x_2(x_2 + x_3)} + \dfrac{x_3^2}{x_3(x_1 + x_2)} \geq$
$\dfrac{(x_2 + x_3)^2}{x_2(x_2 + x_3) + x_3(x_1 + x_2)} = \dfrac{(x_2 + x_3)^2}{x_2^2 + 2x_2x_3 + x_1x_3}$, deci
$\dfrac{(x_2 + x_3)^2}{x_3(x_1x_3 + 2x_2x_3 + x_2^2)} \leq \dfrac{x_2}{x_3(x_2 + x_3)} + \dfrac{1}{x_1 + x_2}$.

Prin urmare,

$$\sum \frac{(x_2 + x_3)^2}{x_3(x_1x_3 + 2x_2x_3 + x_2^2)} \leq \sum \left[\frac{x_2}{x_3(x_2 + x_3)} + \frac{1}{x_1 + x_2}\right] =$$
$$\sum \left[\frac{x_1}{x_2(x_1 + x_2)} + \frac{1}{x_1 + x_2}\right] = \sum \frac{1}{x_2} = \sum \frac{1}{x_1}.$$

Soluţia 2: (*David-Andrei Anghel, Carol-Luca Gasan*)

Notând $S_k = x_k + x_{k+1}$, $k = 1, 2, \ldots, 9$ (indicii sunt luaţi modulo 9), inegalitatea de demonstrat se scrie

$$\sum_{k=1}^{9} \frac{S_k - S_{k+1}}{S_k \cdot S_{k+1} - x_{k+1}(S_k - S_{k+1})} \geq 0.$$

Dar $\dfrac{S_k - S_{k+1}}{S_k \cdot S_{k+1} - x_{k+1}(S_k - S_{k+1})} \geq \dfrac{S_k - S_{k+1}}{S_k \cdot S_{k+1}} \Leftrightarrow x_{k+1}(S_k - S_{k+1})^2 \geq 0$, deci

$$\sum_{k=1}^{9} \frac{S_k - S_{k+1}}{S_k \cdot S_{k+1} - x_{k+1}(S_k - S_{k+1})} \geq \sum_{k=1}^{9} \frac{S_k - S_{k+1}}{S_k \cdot S_{k+1}} = \sum_{k=1}^{9} \left(\frac{1}{S_{k+1}} - \frac{1}{S_k}\right) = 0.$$

Egalitatea are loc dacă $S_1 = S_2 = \ldots = S_9$, adică dacă $x_1 = x_2 = \ldots = x_9$.

O soluţie asemănătoare am primit şi de la *Marius Stănean.*

Problem of the week no. 186
Prove that, for all $x_1, x_2, \ldots, x_9 > 0$, the following inequality holds:
$$\frac{x_1 - x_3}{x_1x_3+2x_2x_3+x_2^2}+\frac{x_2-x_4}{x_2x_4+2x_3x_4+x_3^2}+\ldots+\frac{x_8-x_1}{x_8x_1+2x_9x_1+x_9^2}+\frac{x_9-x_2}{x_9x_2+2x_1x_2+x_1^2}\geq 0.$$

All Russian Mathemtical Olympiad, 2019-2020

Solution 1: (*Orestes Lignos*, taken from https://mathematica.gr)
We have $\dfrac{x_1 - x_3}{x_1x_3 + 2x_2x_3 + x_2^2} = \dfrac{1}{x_3} - \dfrac{(x_2+x_3)^2}{x_3(x_1x_3 + 2x_2x_3 + x_2^2)}$ and its analogues.
The inequality can be written $\sum \dfrac{(x_2+x_3)^2}{x_3(x_1x_3 + 2x_2x_3 + x_2^2)} \leq \sum \dfrac{1}{x_1}$ (1).
From Cauchy-Schwarz, $\dfrac{x_2}{x_2+x_3} + \dfrac{x_3}{x_1+x_2} = \dfrac{x_2^2}{x_2(x_2+x_3)} + \dfrac{x_3^2}{x_3(x_1+x_2)} \geq$
$\dfrac{(x_2+x_3)^2}{x_2(x_2+x_3)+x_3(x_1+x_2)} = \dfrac{(x_2+x_3)^2}{x_2^2+2x_2x_3+x_1x_3}$, hence
$\dfrac{(x_2+x_3)^2}{x_3(x_1x_3+2x_2x_3+x_2^2)} \leq \dfrac{x_2}{x_3(x_2+x_3)} + \dfrac{1}{x_1+x_2}$.
It follows that
$\sum \dfrac{(x_2+x_3)^2}{x_3(x_1x_3+2x_2x_3+x_2^2)} \leq \sum \left[\dfrac{x_2}{x_3(x_2+x_3)} + \dfrac{1}{x_1+x_2}\right] =$
$\sum \left[\dfrac{x_1}{x_2(x_1+x_2)} + \dfrac{1}{x_1+x_2}\right] = \sum \dfrac{1}{x_2} = \sum \dfrac{1}{x_1}$.

Solution 2: (*David-Andrei Anghel, Carol-Luca Gasan*)
Putting $S_k = x_k + x_{k+1}$, $k = 1, 2, \ldots, 9$ (indexes being taken modulo 9), the inequality can be written

$$\sum_{k=1}^{9} \frac{S_k - S_{k+1}}{S_k \cdot S_{k+1} - x_{k+1}(S_k - S_{k+1})} \geq 0.$$

Dar $\dfrac{S_k - S_{k+1}}{S_k \cdot S_{k+1} - x_{k+1}(S_k - S_{k+1})} \geq \dfrac{S_k - S_{k+1}}{S_k \cdot S_{k+1}} \Leftrightarrow x_{k+1}(S_k - S_{k+1})^2 \geq 0$, hence
$\sum_{k=1}^{9} \dfrac{S_k - S_{k+1}}{S_k \cdot S_{k+1} - x_{k+1}(S_k - S_{k+1})} \geq \sum_{k=1}^{9} \dfrac{S_k - S_{k+1}}{S_k \cdot S_{k+1}} = \sum_{k=1}^{9} \left(\dfrac{1}{S_{k+1}} - \dfrac{1}{S_k}\right) = 0.$
Equality holds when $S_1 = S_2 = \ldots = S_9$, i.e. when $x_1 = x_2 = \ldots = x_9$.