Κυπριακη Μαθηματικη Εταιρεια

Β′ Διαγωνισμός Επιλογής κάτω των 15 ½ Ετών

«Ευκλείδης»

Ημερομηνία: 15/02/2020 Ώρα Εξέτασης: 10:00-14:30

Οδηγιες

1. Solve **all** problems, **justifying** fully your answers.
2. Write using blue or black ink. (Figures can be drawn using a pencil)
3. Correction fluid (Tipp-ex) is not permitted.
4. Calculators are not permitted.

Problem 1. Let a,b,c be real numbers such that

$$(x-a)(x-b)(x-c) = x^3 - 10x^2 + 7x + 4$$

for every $x \in \mathbb{R}$. Consider the expressions

$$A = \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} \quad \text{and} \quad B = 100(a^2+b^2+c^2)$$

(i) Prove that $a+b+c = 10$.

(ii) Compute the expressions $ab+bc+ca$ and abc.

(iii) Compare the numbers A and $\frac{1}{B}$.

Problem 2. Prove that for every natural number n, the fraction

$$\frac{n^3(n^6-1)}{504}$$

is an integer number.

Problem 3. Consider the circle (O,ρ) and let A be an external point of the circle. Let (δ) be the line which is perpendicular to the line OA at point A. We draw a line through A which meets the circle at two points, say B and Γ such that B is in-between points A and Γ. The tangents of the circle at points B,Γ, meet the line (δ) at points P,Σ respectively.

(α) Prove that $AP = A\Sigma$.

(β) If the second tangent of the circle through P meets the line $\Sigma\Gamma$ at point M, prove that the line AO passes through M, and it bisects the $\angle \Sigma M P$.

Problem 4. Let Ω be the set $\Omega = \{1,2,3,\ldots,26\}$. Suppose that A is a subset of Ω such that it has the following property:

"There are **no** two distinct elements of A whose difference is a perfect square."

Determine the maximum possible number of elements of the subset A.