TRAINING PROBLEMS FOR THE JBMO - Saudi Arabia, 2017

Problem 1. For each pair of positive integers (x, y) a nonnegative integer $x\Delta y$ is defined. It is known that for all positive integers a and b the following equalities hold:
i. $(a+b)\Delta b = a\Delta b + 1$.
ii. $(a\Delta b)\cdot(b\Delta a) = 0$.
Find the values of the expressions $2016\Delta 121$ and $2016\Delta 144$.

Problem 2. Let ABC be a triangle inscribed in circle (O) such that points B, C are fixed, while A moves on major arc BC of (O). The tangents through B and C to (O) intersect at P. The circle with diameter OP intersects AC and AB at D and E, respectively. Prove that DE is tangent to a fixed circle whose radius is half the radius of (O).

Problem 3. Find all pairs of primes (p, q) such that $p^3 - q^5 = (p+q)^2$.

Problem 4. Let $S = \{-17, -16, \ldots, 16, 17\}$. We call a subset T of S a *good set* if $-x \in T$ for all $x \in T$ and if $x, y, z \in T$ (x, y, z may be equal) then $x + y + z \neq 0$. Find the largest number of elements in a good set.

Problem 5. Let a, b, c be positive real numbers such that $a + b + c = 6$. Prove that

$$\frac{1}{a^2b+16} + \frac{1}{b^2c+16} + \frac{1}{c^2a+16} \geq \frac{1}{8}.$$

Problem 6. Find all pairs of prime numbers (p, q) such that $p^2 + 5pq + 4q^2$ is a perfect square.

Problem 7. Let ABC be a triangle inscribed in te circle (O), with orthocenter H. Let d be an arbitrary line which passes through H and intersects (O) at P and Q. Draw diameter AA' of circle (O). Lines $A'P$ and $A'Q$ meet BC at K and L, respectively. Prove that O, K, L and A' are concyclic.

Problem 8. A chessboard has 64 cells painted black and white in the usual way. A *bishop path* is a sequence of distinct cells such that two consecutive cells have exactly one common point. At least how many bishop paths can the set of all white cells be divided into?

Solution problem 5: The inequality is equivalent to

$$\frac{a^2b}{a^2b+16}+\frac{b^2c}{b^2c+16}+\frac{c^2a}{c^2a+16}\leq 1.$$

But $a^2b+16=a^2b+8+8\geq 3\sqrt[3]{a^2b\cdot 8\cdot 8}=12\sqrt[3]{a^2b}$, therefore it is sufficient to prove that $\sum\frac{a^2b}{12\sqrt[3]{a^2b}}\leq 1$, i.e. $\sum\sqrt[3]{a^4b^2}\leq 12$. This follows from adding $3\sqrt[4]{a^4b^2}\leq a^2+ab+ab$ (AM-GM) with its analogues and using that $(a+b+c)^2=36$.