Problema săptămânii 174
Dacaă $a_1, a_2, \ldots, a_n > 0$, arătaţi că

$$\left(1+\frac{a_1^2}{a_2}\right)\left(1+\frac{a_2^2}{a_3}\right)\cdot\ldots\cdot\left(1+\frac{a_n^2}{a_1}\right) \geq (1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n).$$

Soluţia 1: Conform inegalităţii CBS (forma Titu Andreescu), $\frac{1^2}{1}+\frac{a_1^2}{a_2} \geq \frac{(1+a_1)^2}{1+a_2}$, cu egalitate dacă şi numai dacă $\frac{1}{1}=\frac{a_1}{a_2}$, adică $a_1 = a_2$.
Înmulţind inegalitatea precedentă cu analoagele ei se obţine inegalitatea din enunţ.
Egalitatea are loc atunci când $a_1 = a_2 = \ldots = a_n$.

Soluţia 2: Vom folosi următorul caz particular al inegalităţii lui Hölder (caz particular pe care l-am văzut citat şi drept inegalitatea lui Huygens):
Dacă $m \in \mathbb{N}^*$ şi $x_1, x_2, \ldots, x_m > 0$, atunci are loc inegalitatea

$$(1+x_1)(1+x_2)\cdots(1+x_m) \geq (1+\sqrt[m]{x_1x_2\cdots x_m})^m.$$

(O posibilă demonstraţie: se desfac parantezele şi se aplică inegalitatea mediilor pentru termenii care sunt de acelaşi grad. Atunci se vede că, pentru $n > 1$, egalitatea are loc atunci când numerele sunt egale.)
Vom aplica această inegalitate pentru $m = 2^n - 1$ şi următoarele m numere: $\frac{a_n^2}{a_1}$ şi, pentru fiecare $k = \overline{2, n}$, câte 2^{k-1} termeni egali cu $\frac{a_{n-k+1}^2}{a_{n-k+2}}$. Vom obţine:

$$\left(1+\frac{a_1^2}{a_2}\right)^{2^{n-1}}\left(1+\frac{a_2^2}{a_3}\right)^{2^{n-2}}\cdot\ldots\cdot\left(1+\frac{a_{n-1}^2}{a_n}\right)^2\left(1+\frac{a_n^2}{a_1}\right) \geq (1+\sqrt[m]{x})^m,$$

unde $x = \left(\frac{a_1^2}{a_2}\right)^{2^{n-1}}\left(\frac{a_2^2}{a_3}\right)^{2^{n-2}}\cdot\ldots\cdot\left(\frac{a_{n-1}^2}{a_n}\right)^2\left(\frac{a_n^2}{a_1}\right) = a_1^{2^n-1}$.
Aşadar, am obţinut inegalitatea

$$\left(1+\frac{a_1^2}{a_2}\right)^{2^{n-1}}\left(1+\frac{a_2^2}{a_3}\right)^{2^{n-2}}\cdot\ldots\cdot\left(1+\frac{a_{n-1}^2}{a_n}\right)^2\left(1+\frac{a_n^2}{a_1}\right) \geq (1+a_1)^m.$$

Înmulţind această inegalitate cu alte $n-1$ inegalităţi analoage obţinute prin permutări circulare, fiecare factor de forma $\left(1+\frac{a_k^2}{a_{k+1}}\right)$ va veni la puterea $1+2+\ldots+2^{n-1} = 2^n-1 = m$ astfel că, extrăgând radical de ordin m se obţine tocmai inegalitatea din enunţ.
Egalitatea are loc atunci când $n = 1$ sau $a_1 = a_2 = \ldots = a_n$.

Remarcă: Inegalitatea se poate demonstra şi prin inducţie. Pentru pasul de inducţie trebuie ales $a_{n+1} = \min\{a_1, a_2, \ldots, a_n, a_{n+1}\}$ sau $\max\{a_1, a_2, \ldots, a_n, a_{n+1}\}$.

(Renumerotând indicii, se poate face această presupunere.)

Problem of the week no. 174

If $a_1, a_2, \ldots, a_n$ are positive real numbers, prove that

$$\left(1+\frac{a_1^2}{a_2}\right)\left(1+\frac{a_2^2}{a_3}\right)\cdot\ldots\cdot\left(1+\frac{a_n^2}{a_1}\right) \geq (1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n).$$

Solution 1: According to Titu's Lemma, $\frac{1^2}{1}+\frac{a_1^2}{a_2} \geq \frac{(1+a_1)^2}{1+a_2}$, with equality when $\frac{1}{1}=\frac{a_1}{a_2}$, i.e. $a_1 = a_2$. Multiplying the above inequality with its analogues gives the desired inequality. Equality holds when $a_1 = a_2 = \ldots = a_n$.

Solution 2: We use the following variant of Hölder's inequality (a variant sometimes called Huygens' inequality):

If $m \in \mathbb{N}$ and $x_1, x_2, \ldots, x_m > 0$, then the following inequality holds:

$$(1+x_1)(1+x_2)\cdots(1+x_m) \geq (1+\sqrt[m]{x_1 x_2 \cdots x_m})^m.$$

(A possible proof: expand the LH side and apply the AM-GM inequality for groups of all the terms that have the same degree. From this it is easy to see that, for $n > 1$, equality holds when the numbers are all equal.)

We apply this inequality for $m = 2^n - 1$ and the following m numbers: $\frac{a_n^2}{a_1}$ and, for every $k = \overline{2,n}$, we take 2^{k-1} terms equal to $\frac{a_{n-k+1}^2}{a_{n-k+2}}$. We obtain:

$$\left(1+\frac{a_1^2}{a_2}\right)^{2^{n-1}}\left(1+\frac{a_2^2}{a_3}\right)^{2^{n-2}}\cdot\ldots\cdot\left(1+\frac{a_{n-1}^2}{a_n}\right)^2\left(1+\frac{a_n^2}{a_1}\right) \geq (1+\sqrt[m]{x})^m,$$

where $x = \left(\frac{a_1^2}{a_2}\right)^{2^{n-1}}\left(\frac{a_2^2}{a_3}\right)^{2^{n-2}}\cdot\ldots\cdot\left(\frac{a_{n-1}^2}{a_n}\right)^2\left(\frac{a_n^2}{a_1}\right) = a_1^{2^n-1}$.

Thus, we got $\left(1+\frac{a_1^2}{a_2}\right)^{2^{n-1}}\left(1+\frac{a_2^2}{a_3}\right)^{2^{n-2}}\cdot\ldots\cdot\left(1+\frac{a_{n-1}^2}{a_n}\right)^2\left(1+\frac{a_n^2}{a_1}\right) \geq (1+a_1)^m$.

Multiplying this inequality with $n-1$ similar inequalities obtained from this one by circular permutation, each factor of the type $\left(1+\frac{a_k^2}{a_{k+1}}\right)$ will appear at the power $1+2+\ldots+2^{n-1} = 2^n-1 = m$ therefore, drawing the root of order m we get the desired inequality.

Equality holds for $n = 1$ or $a_1 = a_2 = \ldots = a_n$.