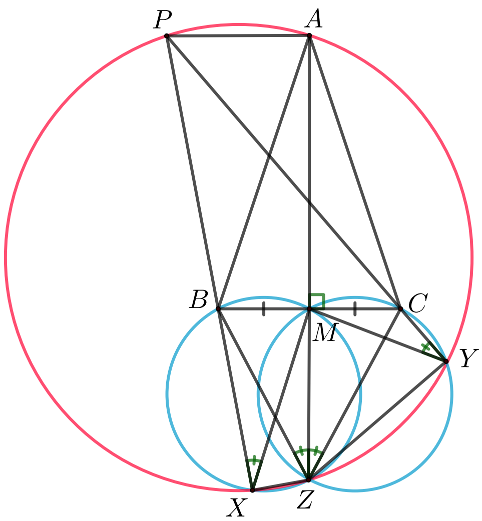
Problema săptămânii 169

Fie ABC un triunghi cu $AB = AC$ şi fie M mijlocul laturii BC. Fie P un punct pentru care $PB < PC$ şi PA este paralelă cu BC. Pe dreptele PB şi PC se consideră punctele X, respectiv Y astfel încât $B \in (PX)$, $C \in (PY)$ şi $\sphericalangle PXM \equiv \sphericalangle PYM$. Arătaţi că patrulaterul $APXY$ este inscriptibil.

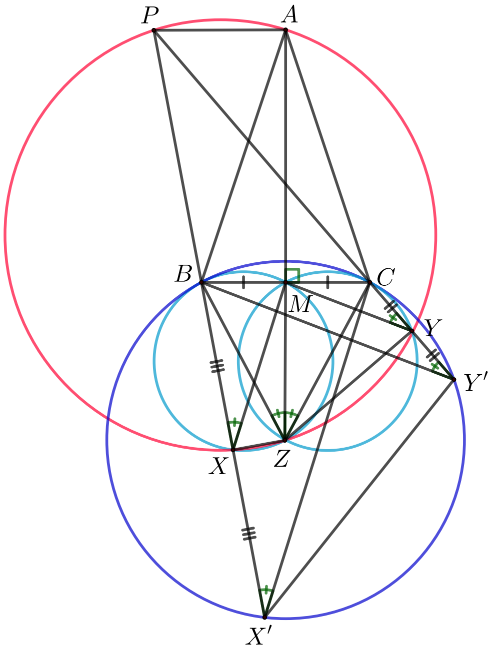
Olimpiadă Elveţia, 2019

Soluţie: Fie Z al doilea punct de intersecţie a cercurilor circumscrise triunghiurilor MBX şi MCY. Atunci $\sphericalangle MZB \equiv \sphericalangle MXB \equiv \sphericalangle MYC \equiv \sphericalangle MZC$, deci, în triunghiul BZC, ZM este mediană şi bisectoare. Rezultă că Z se află pe mediatoarea lui $[BC]$, adică pe AM. Deducem că $m(\sphericalangle ZXP) = m(\sphericalangle ZXB) = m(\sphericalangle ZMB) = 90°$ şi $m(\sphericalangle ZYP) = m(\sphericalangle ZYC) = m(\sphericalangle ZMC) = 90°$. De asemenea, $m(\sphericalangle ZAP) = 90°$, deci punctele X, Y şi A se află pe cercul de diametru $[PZ]$. Prin urmare, $APXY$ este inscriptibil.



Comentariu: Cum se putea „ghici" ideea de a considera punctul Z din rezolvarea de mai sus?

Ipoteza $\sphericalangle PXM \equiv \sphericalangle PYM$ şi concluzia sugerează să traducem egalitatea de unghiuri în condiţia de inscriptibilitate a unui patrulater. Dacă X' şi Y' sunt simetricele punctelor B şi C faţă de X, respectiv Y, atunci $\sphericalangle PX'C \equiv \sphericalangle PXM \equiv \sphericalangle PYM \equiv \sphericalangle PY'B$, adică $BX'Y'C$ este inscriptibil. Deducem că mediatoarele laturilor sale sunt concurente într-un punct Z, deci punctele A, X, Y se găsesc, toate, pe cercul de diametrul $[PZ]$.



Remarcă: Concluzia problemei rămâne valabilă şi în cazul în care $X \in (PB)$, $Y \in (PC)$ satisfac $\sphericalangle PXM \equiv \sphericalangle PYM$.

Am primit şi o soluţie interesantă, cu inversiune de centru P, de la *Luca Pană*.

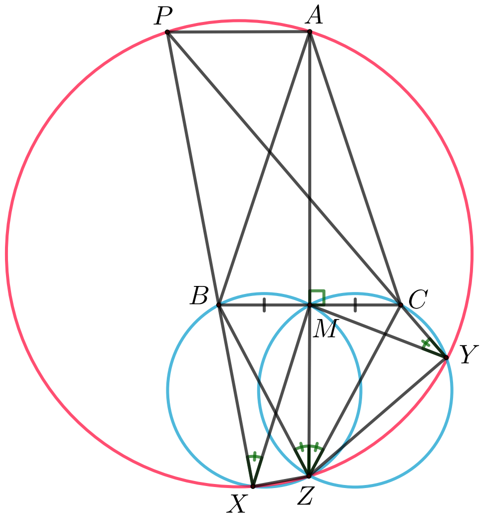
Problem of the week no. 169

Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. On the lines PB and PC consider points X and Y, respectively, such that $B \in (PX)$, $C \in (PY)$ and $\sphericalangle PXM = \sphericalangle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

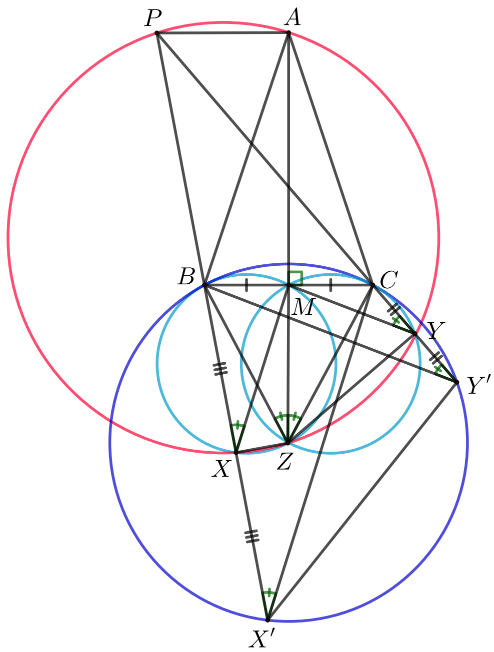
Swiss Mathematical Olympiad, 2019

Solution:

Let Z be the second intersection point of the circumcircles of triangles MBX and MCY. Then $\sphericalangle MZB = \sphericalangle MXB = \sphericalangle MYC = \sphericalangle MZC$, which means that in triangle BZC, ZM is both the median and the angle bisector. It follows that Z is on the perpendicular bisector of $[BC]$, that is, on AM. It follows that $\sphericalangle ZXP = \sphericalangle ZXB = \sphericalangle ZMB = 90^\circ$ and $\sphericalangle ZYP = \sphericalangle ZYC = \sphericalangle ZMC = 90^\circ$. Also, $m(\sphericalangle ZAP) = 90^\circ$, therefore points X, Y and A lie on the circle of diameter $[PZ]$. In conclusion, $APXY$ is cyclic.



Comment: How could one "guess" to consider the point Z?
The hypothesis $\sphericalangle PXM = \sphericalangle PYM$ and the conclusion suggest translating the equality of the angles into the condition that a certain quadrilateral is cyclic. If X' and Y' are the reflections of B and C with respect to X and Y, respectively, then $\sphericalangle PX'C = \sphericalangle PXM = \sphericalangle PYM = \sphericalangle PY'B$, i.e. $BX'Y'C$ is cyclic. It follows that the perpendicular bisectors of its sides are concurrent at some point Z, which means that points A, X, Y lie on the circle of diameter $[PZ]$.



Remark: The conclusion of the problem remains valid even in the case when $X \in (PB)$, $Y \in (PC)$ satisfy $\sphericalangle PXM = \sphericalangle PYM$.