Problema săptămânii 167
Demonstraţi că există o infinitate de numere naturale n pentru care suma cifrelor lui 2^n este mai mare decât suma cifrelor lui 2^{n+1}.

Serghei Koneaghin (*Kvant*, problema 390)

Soluţie:
I. Resturile împărţirii lui 2^n la 9 sunt, periodic, $1, 2, 4, 8, 7, 5$.
II. Deoarece $2^{3k} < 10^k$, avem chiar $2^{3k+2} < 4 \cdot 10^k$, deci 2^n are cel mult $\dfrac{n}{3} + 1$ cifre.
Presupunând că am avea numai un număr finit de numere naturale n pentru care $s(2^n) > s(2^{n+1})$, ar trebui să existe un N astfel încât $s(2^n) \leq s(2^{n+1})$, $\forall n \geq N$.
Dar atunci, pentru $6k \geq N$, am avea, conform I:

$$s(2^{6k+6}) - s(2^{6k+5}) = (M_9 + 1) - (M_9 + 5) = M_9 + 5 \geq 5,$$

$$s(2^{6k+5}) - s(2^{6k+4}) = (M_9 + 5) - (M_9 + 7) = M_9 + 7 \geq 7,$$
$$s(2^{6k+4}) - s(2^{6k+3}) = (M_9 + 7) - (M_9 + 8) = M_9 + 8 \geq 8,$$
$$s(2^{6k+3}) - s(2^{6k+2}) = (M_9 + 8) - (M_9 + 4) = M_9 + 4 \geq 4,$$
$$s(2^{6k+2}) - s(2^{6k+1}) = (M_9 + 4) - (M_9 + 2) = M_9 + 2 \geq 2,$$
$$s(2^{6k+1}) - s(2^{6k}) = (M_9 + 2) - (M_9 + 1) = M_9 + 1 \geq 1.$$

Adunând aceste relaţii, obţinem că $s(2^{6k+6}) - s(2^{6k}) \geq 27$, deci $s(2^{6k+6m}) - s(2^{6k}) \geq 27m$, $\forall m \in \mathbb{N}$. Dar 2^{6k+6m}, având cel mult $2k + 2m + 1$ cifre, are suma cifrelor cel mult $9(2k + 2m + 1)$, deci trebuie să avem $9(2k + 2m + 1) \geq s(2^{6k}) + 27m$, adică $9m \leq 18k + 9 - s(2^{6k})$ pentru orice m, ceea ce nu se poate.
Contradicţia obţinută arată că există o infinitate de numere n pentru care $s(2^n) > s(2^{n+1})$.

Problem of the week no. 167
Prove that there are infinitely many positive integers n for which the sum of the digits of 2^n is larger than the sum of the digits of 2^{n+1}.

S. Konyagin (*Quantum Magazine*, problem 390)

Solution:
I. The remainders of 2^n when divided by 9 are, periodically, $1, 2, 4, 8, 7, 5$.
II. Since $2^{3k} < 10^k$, and even $2^{3k+2} < 4 \cdot 10^k$, we see that 2^n has at most $\dfrac{n}{3} + 1$ digits.
Assuming there is only a finit number of positive integers n that satisfy $s(2^n) > s(2^{n+1})$, there must be an N such that $s(2^n) \leq s(2^{n+1})$, $\forall n \geq N$.
But then, for $6k \geq N$, according to I, we would have:

$$s(2^{6k+6}) - s(2^{6k+5}) = (M_9 + 1) - (M_9 + 5) = M_9 + 5 \geq 5,$$

$$s(2^{6k+5}) - s(2^{6k+4}) = (M_9 + 5) - (M_9 + 7) = M_9 + 7 \geq 7,$$

$$s(2^{6k+4}) - s(2^{6k+3}) = (M_9 + 7) - (M_9 + 8) = M_9 + 8 \geq 8,$$
$$s(2^{6k+3}) - s(2^{6k+2}) = (M_9 + 8) - (M_9 + 4) = M_9 + 4 \geq 4,$$
$$s(2^{6k+2}) - s(2^{6k+1}) = (M_9 + 4) - (M_9 + 2) = M_9 + 2 \geq 2,$$
$$s(2^{6k+1}) - s(2^{6k}) = (M_9 + 2) - (M_9 + 1) = M_9 + 1 \geq 1.$$

Adding up these relations, we get $s(2^{6k+6}) - s(2^{6k}) \geq 27$, hence $s(2^{6k+6m}) - s(2^{6k}) \geq 27m$, $\forall\, m \in \mathbb{N}$. But 2^{6k+6m} has at most $2k + 2m + 1$ digits, so the sum of its digits is at most $9(2k + 2m + 1)$, so we should have $9(2k + 2m + 1) \geq s(2^{6k}) + 27m$, i.e. $9m \leq 18k + 9 - s(2^{6k})$ for all m, which is not possible.
The contradiction we have obtained shows that there are infinitely many n for which $s(2^n) > s(2^{n+1})$.