Problema săptămânii 134
Dacă n este un număr natural nenul, iar $\{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$, demonstraţi că

$$1^{a_1} + 2^{a_2} + \ldots + n^{a_n} \leq 1^1 + 2^2 + \ldots + n^n.$$

Concursul Traian Lalescu, 2003

Soluţie:
Vom demonstra afirmaţia prin inducţie după n. Pentru $n = 1$ nu avem ce demonstra. Presupunem afirmaţia adevărată pentru n şi o demonstrăm pentru $n + 1$. Presupunând că $\{a_1, a_2, \ldots, a_n, a_{n+1}\} = \{1, 2, \ldots, n, n + 1\}$, există (un unic) $k \in \{1, 2, \ldots, n + 1\}$ astfel încât $a_k = n + 1$.
Dacă $k = n+1$, avem $1^{a_1}+2^{a_2}+\ldots+n^{a_n}+(n+1)^{n+1} \leq 1^1+2^2+\ldots+n^n+(n+1)^{n+1}$ în baza ipotezei de inducţie aplicate numerelor $a_1, a_2, \ldots, a_n$.
Dacă $k \in \{1, 2, \ldots, n\}$, atunci $k^{a_k}+(n+1)^{a_{n+1}} \leq k^{a_{n+1}}+(n+1)^{n+1} \Leftrightarrow k^{n+1}-k^{a_{n+1}} \leq (n+1)^{n+1}-(n+1)^{a_{n+1}} \Leftrightarrow k^{a_{n+1}}(k^{n+1-a_{n+1}} - 1) \leq (n+1)^{a_{n+1}}((n + 1)^{n+1-a_{n+1}} - 1)$, inegalitate evidentă deoarece $1 \leq k \leq n + 1$ şi $a_{n+1} \leq n + 1$. Rezultă că $1^{a_1} + 2^{a_2} + \ldots (k-1)^{a_{k-1}}+k^{a_k}+(k+1)^{a_{k+1}}+\ldots+n^{a_n}+(n+1)^{a_{n+1}} \leq 1^{a_1}+2^{a_2}+\ldots+(k-1)^{a_{k-1}}+k^{a_{n+1}}+(k+1)^{a_{k+1}}+\ldots+n^{a_n}+(n+1)^{n+1} \leq 1^1+2^2+\ldots+n^n+(n+1)^{n+1}$ în baza ipotezei de inducţie aplicate numerelor $\{a_1, a_2, \ldots, a_{k-1}, a_{n+1}, a_k, \ldots, a_n\} = \{1, 2, \ldots, n\}$.
Cu aceasta, inducţia este încheiată, iar afirmaţia din enunţ demonstrată.

Problem of the week no. 134
If n is a positive integer, and $\{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$, prove that

$$1^{a_1} + 2^{a_2} + \ldots + n^{a_n} \leq 1^1 + 2^2 + \ldots + n^n.$$

Traian Lalescu Contest, 2003

Solution:
We prove the statement by induction after n. For $n = 1$ there is nothing to prove. Assume the statement to be true for a value of n and let us prove it for $n + 1$. Assuming that $\{a_1, a_2, \ldots, a_n, a_{n+1}\} = \{1, 2, \ldots, n, n + 1\}$, there exists a unique $k \in \{1, 2, \ldots, n + 1\}$ such that $a_k = n + 1$.
If $k = n + 1$, then $1^{a_1} + 2^{a_2} + \ldots + n^{a_n} + (n + 1)^{n+1} \leq 1^1 + 2^2 + \ldots + n^n + (n + 1)^{n+1}$ according to the inductive hypothesis for the numbers $a_1, a_2, \ldots, a_n$.
If $k \in \{1, 2, \ldots, n\}$, then $k^{a_k} + (n + 1)^{a_{n+1}} \leq k^{a_{n+1}} + (n + 1)^{n+1} \Leftrightarrow k^{n+1} - k^{a_{n+1}} \leq (n+1)^{n+1}-(n+1)^{a_{n+1}} \Leftrightarrow k^{a_{n+1}}(k^{n+1-a_{n+1}} - 1) \leq (n+1)^{a_{n+1}}((n + 1)^{n+1-a_{n+1}} - 1)$, which is obvious because $1 \leq k \leq n + 1$ and $a_{n+1} \leq n + 1$. It follows that $1^{a_1} + 2^{a_2} + \ldots (k - 1)^{a_{k-1}} + k^{a_k} + (k + 1)^{a_{k+1}} + \ldots + n^{a_n} + (n + 1)^{a_{n+1}} \leq 1^{a_1} + 2^{a_2} + \ldots + (k - 1)^{a_{k-1}} + k^{a_{n+1}} + (k + 1)^{a_{k+1}} + \ldots + n^{a_n} + (n + 1)^{n+1} \leq 1^1 + 2^2 + \ldots + n^n + (n + 1)^{n+1}$ according to the inductive hypothesis applied for the numbers $\{a_1, a_2, \ldots, a_{k-1}, a_{n+1}, a_k, \ldots, a_n\} = \{1, 2, \ldots, n\}$.
Thus, the proof by induction is finished; the statement is true for all $n \geq 1$.